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ASYMPTOTIC EXPANSIONS FOR THE POWER OF DISTRIBUTIONFREE TESTS IN THE TWO-SAMPLE PROBLEM*)

by

P.J. Bickel**) & W.R. van Zwet***)

ABSTRACT

Asymptotic expansions are established for the power of distributionfree tests in the two-sample problem. These expansions are then used to obtain deficiencies in the sense of HODGES and LEHMANN (1970) for distributionfree tests with respect to their parametric competitors and for the estimators of shift associated with these tests.

KEY WORDS & PHRASES: *Distributionfree tests, linear rank tests, permutation test, Hodges-Lehmann estimators, power, contiguous alternatives, asymptotic expansions, Edgeworth expansions, deficiency, rejective sampling, sampling without replacement.*

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1. INTRODUCTION

Let $X_1, X_2, \ldots, X_N$, $N = m+n$, be independent random variables such that $X_1, \ldots, X_m$ are identically distributed with common distribution function F and density f and $X_{m+1}, \ldots, X_N$ are identically distributed with distribution function G and density g. For $N = 2,3,\ldots$ and $0 < \varepsilon \leq m/N \leq 1-\varepsilon < 1$, consider the problem of testing the hypothesis $F = G$ against a sequence of alternatives that is contiguous to the hypothesis. The level α of the sequence of tests is fixed in $(0,1)$. Standard tests for this two-sample problem are linear rank tests and permutation tests and expressions for the limiting powers of such tests are well-known. In this paper we shall establish asymptotic expansions to order N^{-1} for the powers π_N of such tests, i.e. expressions of the form $\pi_N = c_0 + c_1 N^{-\frac{1}{2}} + c_{2,N} N^{-1} + o(N^{-1})$. Of course this involves finding similar expansions for the distribution function of the test statistic under the hypothesis as well as under contiguous alternatives. For simplicity we shall eventually limit our discussion to contiguous location alternatives. Extension of the results to general contiguous alternatives is straightforward but messy.

A number of authors have computed formal expansions for the distributions of various two-sample rank statistics without proof of their validity. Their purpose was to obtain better numerical approximations for the critical value of the test statistic and the power of the test than can be provided by the usual normal approximation. For an account of this work we refer to a review paper of BICKEL (1974), which incidentally also contains a short preview of the present paper including a brief description of the expansion of the distribution function of the two-sample linear rank statistic under the hypothesis (c.f. corollary 2.1 in the present paper). This result was also proved independently by ROBINSON (1977). An earlier proof by ROGERS (1971) for the special case of the two-sample Wilcoxon statistic under the hypothesis unfortunately appears to contain a non-trivial error.

We shall not discuss the numerical aspects of the expansions we obtain but we shall concentrate on a rather delicate type of asymptotic comparison of the power functions of various parametric and nonparametric tests. Consider two sequences of tests $\{T_N\}$ and $\{T'_N\}$ for the same hypothesis at the same fixed level α. Let $\pi_N(\theta_N)$ and $\pi'_N(\theta_N)$ denote the powers of these

tests against the same sequence of contiguous alternatives parametrized by a parameter θ. If T_N is more powerful than T'_N we search for a number $k_N = N + d_N$ such that $\pi_N(\theta_N) = \pi'_{k_N}(\theta_N)$. Here k_N and d_N are treated as continuous variables, the power π'_N being defined for real N by linear interpolation between consecutive integers. The quantity d_N was named the deficiency of $\{T'_N\}$ with respect to T_N by HODGES and LEHMANN (1970), who introduced this concept and initiated its study. Of course, in many cases of interest d_N is analytically intractable and one can only study its asymptotic behavior as N tends to infinity.

Suppose that for $N \to \infty$, the ratio N/k_N tends to a limit e, the asymptotic relative efficiency of $\{T'_N\}$ with respect to $\{T_N\}$. If $0 < e < 1$, we have $d_N \sim (e^{-1} - 1)N$ and further asymptotic information about d_N is not particularly revealing. On the other hand, if $e = 1$, the asymptotic behavior of d_N - which may now be anything from $o(1)$ to $o(N)$ - does provide important additional information. Of special interest is the case where d_N tends to a finite limit.

Asymptotic expansions for the power of the type we discussed above are precisely what is needed for an asymptotic evaluation of d_N. With the aid of such expansions we arrive at the following results. Let F be a distribution function with density f, let b be a positive real number and define $\theta_N = bN^{-\frac{1}{2}}$. Consider the problem of testing the hypothesis (F,F) against the sequence of simple alternatives $(F(\cdot+\Delta_N\theta_N), F(\cdot-(1-\Delta_N)\theta_N))$ at level α. Let d_N denote the deficiency of the locally most powerful rank test with respect to the most powerful test for this problem. For the rank test the power is independent of Δ_N but for the most powerful test it is not and we choose Δ_N in such a way that the power of the most powerful test is minimal. Under certain regularity conditions on F we establish an expansion for d_N with remainder $o(1)$. To indicate the qualitative behavior of d_N it suffices to note that the expansion is of the form

$$(1.1) \qquad d_N = \frac{1}{\int \psi_1^2(t)dt} \sum_{j=1}^{N} \sigma^2(\psi_1(U_{j:N})) + \tilde{d}_{N,0} + o(1)$$

where $\psi_1 = f'(F^{-1})/f(F^{-1})$, σ^2 indicates a variance, $U_{j:N}$ denotes the j-th order statistic of a sample of size N from a uniform distribution on (0,1)

and $\tilde{d}_{N,0} = O(1)$. Alternatively we may write

$$(1.2) \qquad d_N = \frac{1}{\int \psi_1^2(t)dt} \int_{N^{-1}}^{1-N^{-1}} (\psi_1'(t))^2 t(1-t)dt + \tilde{d}_{N,0} + o(1)$$
$$+ O\left(N^{-\frac{1}{2}} \int_{N^{-1}}^{1-N^{-1}} (\psi_1'(t))^2 \{t(1-t)\}^{\frac{1}{2}} dt\right),$$

where ψ_1' is the derivative of ψ_1. If we replace the exact scores $-E\psi_1(U_{j:N})$ in the locally most powerful rank test by the corresponding approximate scores $-\psi_1(j/(N+1))$, then (1.1) changes to

$$(1.3) \qquad d_N = \frac{1}{\int \psi_1^2(t)dt} \sum_{j=1}^{N} E\{\psi_1(U_{j:N}) - \psi_1(j/(N+1))\}^2 + \tilde{d}_{N,0} + o(1)$$

and (1.2) continues to hold. Thus the asymptotic behavior of d_N is governed by that of the first term in these expansions and under the conditions imposed, all we can say is that it is $o(N^{1/3})$ but not $o(1)$. Typically, however, it will be $O(1)$ or only slightly larger than that. By taking F to be a normal distribution we find that the deficiency of both the normal scores test and van der Waerden's test with respect to the test based on the difference of the sample means for contiguous normal location alternatives is asymptotic to $\log \log N$. For logistic shift alternatives the deficiency of Wilcoxon's test with respect to the most powerful parametric test tends to a finite limit. Turning to distributionfree tests other than rank tests, we find that for contiguous normal location alternatives the deficiency of the permutation test based on the sample means with respect to Student's test tends to zero for $N \to \infty$.

If the locally most powerful rank test for shift has nondecreasing scores, then there exists a corresponding Hodges-Lehmann estimator of shift in the two-sample problem (c.f. HODGES and LEHMANN (1963)). There is a similar correspondence between the locally most powerful parametric test for shift and the maximum likelihood estimator of shift in the two-sample problem. We shall exploit this correspondence to obtain asymptotic expansions for the distribution functions of these estimators. We shall show that, when suitably defined, the deficiency of the Hodges-Lehmann

estimator associated with the locally most powerful rank test with respect to the maximum likelihood estimator is asymptotically equivalent to the deficiency of the parent tests for $\alpha = \frac{1}{2}$.

This paper is thus the natural counterpart of ALBERS, BICKEL and VAN ZWET (ABZ) (1976) where exactly the same programme is carried out for the one-sample problem. Without exception the results are also qualitatively the same but contrary to what one might think at first sight, this in itself is rather surprising. Of course there is a strong similarity between the one- and two-sample cases but there is also one major difference. In the nonparametric one-sample location problem the underlying distribution is always symmetric both under the hypothesis and under the alternative. Because of this symmetry, the power expansions for contiguous location alternatives donot contain a term of order $N^{-\frac{1}{2}}$ for any of the parametric or nonparametric tests considered. Since attention is restricted to sequences of tests $\{T_N\}$ and $\{T'_N\}$ with asymptotic relative efficiency 1, the leading terms of the power expansions coincide and these expansions must therefore be of the form $\pi_N = c_0 + c_{2,N}N^{-1} + o(N^{-1})$ and $\pi'_N = c_0 + c'_{2,N}N^{-1} + o(N^{-1})$. In the comparison of rank tests T'_N with parametric tests T_N it is found that the deficiency d_N is of the order of $N(\pi_N - \pi'_N) = (c_{2,N} - c'_{2,N}) + o(1) = o(N^{1/3})$. In the two-sample problem, however, the underlying distributions are not required to be symmetric and as a result the power expansions do in general contain a term of order $N^{-\frac{1}{2}}$. It is not clear a priori that this term should be the same in each expansion and because d_N is again of the order of $N(\pi_N - \pi'_N)$, one should expect d_N to be of the order $N^{\frac{1}{2}}$. It turns out, however, that for the most powerful test, the locally most powerful test, the locally most powerful rank test and its approximate scores analogue, the term of order $N^{-\frac{1}{2}}$ in the power expansion for contiguous location alternatives is in fact the same for each of these four tests. Borrowing a phrase from PFANZAGL (1977) who noted the same phenomenon for the (asymmetric) parametric one-sample problem, first order efficiency apparently implies second order efficiency in these cases. It follows that again d_N is of the order of $(c_{2,N} - c'_{2,N})$ and since $c_{2,N}$ and $c'_{2,N}$ exhibit precisely the same asymptotic behavior as in the one-sample case, our deficiency results are qualitatively the same as in ABZ (1976). The reader should note that Pfanzagl's concept of second order efficiency which in

general implies $d_N = o(N^{\frac{1}{2}})$, is different from Rao's concept of second order efficiency as discussed in EFRON (1975), which is more in the nature of $d_N = o(1)$. This difference in terminology is not as illogical as it may seem because Rao's concept is related to the asymptotic performance of an estimator M_N as measured by the asymptotic variance of $N^{\frac{1}{2}} M_N$ and expansions for this quantity are typically in powers of N^{-1} rather than $N^{-\frac{1}{2}}$.

Throughout this paper we shall draw heavily on the techniques developped for the one-sample case in ABZ (1976) but several new difficulties appear that make the two-sample case essentially more complicated. The main source of trouble is the occurrence of terms of order $N^{-\frac{1}{2}}$ in our expansions. Not only do they make the actual computation of the expansions much more laborious, but their presence also poses a number of technical problems that are hard to handle under the conditions imposed, which are comparable to those in ABZ (1976). Another complicating factor is that the distribution theory for the two-sample rank statistic is more involved than for its one-sample counterpart. In the one-sample case a conditioning argument reduces the rank statistic to a weighted sum of independent Bernoulli random variables. A similar argument in the two-sample case leads to the much less manageable random variable indicated below.

In section 2 we point out that for arbitrary F and G, the conditional distribution of the two-sample linear rank statistic given the order statistics of the combined sample is the same as the distribution of the sample sum in a rejective sampling scheme. We establish an expansion for the distribution function of such a sample sum which may be of interest in its own right. As a corollary we obtain an expansion for the distribution function of the rank statistic under the hypothesis. In section 3 we return to general F and G and obtain an unconditional expansion for the distribution function of the rank statistic. We specialize to contiguous location alternatives in section 4 and derive an expansion for the power of the rank test. In section 5 we deal with the important case where the scores are exact or approximate scores generated by a smooth function. The permutation test based on the sample means is discussed in section 6. The results on deficiencies of distributionfree tests are contained in section 7. Section 8 is devoted to estimators. Some technical results are dealt with in the appendix.

2. AN EXPANSION FOR THE CONDITIONAL DISTRIBUTION OF TWO-SAMPLE RANK STATISTICS AND ITS APPLICATION TO REJECTIVE SAMPLING

Let $X_1,X_2,\ldots,X_N$, $N = m + n$, be independent random variables (r.v.'s) such that $X_1,\ldots,X_m$ are identically distributed (i.d.) with common distribution function (d.f.) F and density f and $X_{m+1},\ldots,X_N$ are i.d. with common d.f. G and density g. Let $Z_1 < Z_2 < \ldots < Z_N$ denote the order statistics of $X_1,\ldots,X_N$, define the antiranks $D_1,D_2,\ldots,D_N$ by $X_{D_j} = Z_j$ and let

$$V_j = \begin{cases} 1 & \text{if } m+1 \le D_j \le N \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

For a specified vector of scores $a = (a_1,a_2,\ldots,a_N)$ define a two-sample rank statistic by

$$T = \sum_{j=1}^{N} a_j V_j. \tag{2.2}$$

Our aim is to obtain an asymptotic expansion as $N \to \infty$ for the distribution of T for suitable sequences of pairs of d.f.'s (F_N,G_N), arrays of scores $\{a_{j,N}\}$, $1 \le j \le N$, and sample sizes (m_N,n_N). As in ALBERS, BICKEL and VAN ZWET (ABZ) (1976) we shall suppress dependence on N whenever possible and formally present our results in terms of error bounds for fixed, but arbitrary, values of N.

Under the null-hypothesis that F = G,

$$P(V_1=v_1,\ldots,V_N=v_N) = \frac{1}{\binom{N}{n}}$$

for any vector $(v_1,\ldots,v_N)$ with m co-ordinates equal to 0 and n co-ordinates equal to 1. In general, conditional on $Z = (Z_1,\ldots,Z_N)$,

$$P(V_1=v_1,\ldots,V_N=v_N \mid Z) = c^{-1}(P) \prod_{j=1}^{N} P_j^{v_j}(1-P_j)^{1-v_j}, \tag{2.3}$$

where

$$P_j = \frac{\lambda g(Z_j)}{(1-\lambda)f(Z_j) + \lambda g(Z_j)}, \tag{2.4}$$

$$\lambda = \frac{n}{N} \quad , \tag{2.5}$$

$$c(P) = \sum \prod_{j=1}^{N} P_j^{w_j}(1-P_j)^{1-w_j} \quad , \tag{2.6}$$

and the summation is over all vectors $(w_1,\ldots,w_N)$ consisting of m zeros and n ones.

Let $W_1,W_2,\ldots,W_N$ be independent r.v.'s with $P(W_j=1) = 1 - P(W_j=0) = p_j$, $1 \le j \le N$. Suppose that

$$\begin{aligned} p_j &= 0 \qquad \text{for at most m indices j} \\ p_j &= 1 \qquad \text{for at most n indices j} \end{aligned} \tag{2.7}$$

and consider the conditional distribution of $\sum a_j W_j$ given that $\sum W_j = n$. Note that if we replace $p = (p_1,\ldots,p_N)$ by $P = (P_1,\ldots,P_N)$, then this is the distribution of T given Z. For general p this distribution is of interest in its own right since $\sum a_j W_j$ given $\sum W_j = n$ is the sample sum we obtain when we use a rejective sampling scheme with parameters $p_1,\ldots,p_N$ in selecting a sample of size n from the sampling frame $\{a_1,a_2,\ldots,a_N\}$ (see HÁJEK (1964) for details).

Define

$$\rho(t,p) = E\left(\exp\{itN^{-\frac{1}{2}} \sum_{j=1}^{N} a_j(W_j-p_j)\} \;\middle|\; \sum_{j=1}^{N} W_j=n\right), \tag{2.8}$$

$$R(x,p) = P\left(N^{-\frac{1}{2}} \sum_{j=1}^{N} a_j(W_j-p_j) \le x \;\middle|\; \sum_{j=1}^{N} W_j=n\right). \tag{2.9}$$

Our program for obtaining an Edgeworth expansion for the d.f. of T parallels in part that of ABZ (1976). We obtain a formula for ρ. From this formula we obtain an expansion for ρ which we can rigorously translate into an Edgeworth expansion for R. Because of the connection with rejective sampling we isolate this result as the only theorem in this section. In the next section we proceed with our main program and obtain an expansion for the d.f. of T by replacing p by P and taking the expectation of the resulting expression. We begin with

LEMMA 2.1. *Define*

$$\psi(s,t,p) = \exp\{isN^{-\frac{1}{2}} \sum_{j=1}^{N} (p_j-\lambda)\} \prod_{j=1}^{N} [p_j \exp\{iN^{-\frac{1}{2}}(1-p_j)(s+a_jt)\} \tag{2.10}$$

$$+ (1-p_j)\exp\{- iN^{-\frac{1}{2}}p_j(s+a_jt)\}],$$

$$\nu(t,p) = \int_{-\pi N^{\frac{1}{2}}}^{\pi N^{\frac{1}{2}}} \psi(s,t,p)ds, \tag{2.11}$$

$$c(p) = \sum \prod_{j=1}^{N} p_j^{w_j}(1-p_j)^{1-w_j}, \tag{2.12}$$

where the last summation is over all vectors $(w_1,\ldots,w_N)$ *consisting of* m *zeros and* n *ones. Then, if* (2.7) *is satisfied,*

$$\rho(t,p) = \frac{1}{2\pi c(p)N^{\frac{1}{2}}} \int_{-\pi N^{\frac{1}{2}}}^{\pi N^{\frac{1}{2}}} \psi(s,t,p)ds = \frac{\nu(t,p)}{\nu(0,p)} . \tag{2.13}$$

PROOF. Begin with the identity

$$E(\exp\{iN^{-\frac{1}{2}}[s \sum(W_j-p_j) + t \sum a_j(W_j-p_j)]\})$$

$$= \sum_{k=0}^{N} E(\exp\{itN^{-\frac{1}{2}}\sum a_j(W_j-p_j)\}|\sum W_j=k)P(\sum W_j=k)\exp\{isN^{-\frac{1}{2}}(k-\sum p_j)\}.$$

Because the system $\{(2\pi N^{\frac{1}{2}})^{-\frac{1}{2}} \exp(iksN^{-\frac{1}{2}})\colon k = 0, \pm 1,\ldots\}$ is orthonormal on $[-\pi N^{\frac{1}{2}}, \pi N^{\frac{1}{2}}]$ this implies

$$\rho(t,p) = (2\pi N^{\frac{1}{2}}P(\sum W_j=n))^{-1} \int_{-\pi N^{\frac{1}{2}}}^{\pi N^{\frac{1}{2}}} \exp\{isN^{-\frac{1}{2}}\sum(p_j-\lambda)\}$$

$$\times E(\exp\{iN^{-\frac{1}{2}}\sum(s+a_jt)(W_j-p_j)\})ds.$$

Elementary considerations now yield (2.13). □

Note that if $p_j = \lambda$ for all j - which corresponds to the null-hypothesis in the two-sample problem - our formula agrees with that of ERDÖS and RÉNYI for random sampling without replacement (cf. RÉNYI (1970) p. 462).

In fact their result motivated our approach.

In our asymptotic study of ψ, ν and ρ we shall repeatedly come across the following functions of p.

$$\omega(p) = N^{-\frac{1}{2}} \sum_{j=1}^{N} (p_j - \lambda), \tag{2.14}$$

$$\sigma^2(p) = N^{-1} \sum_{j=1}^{N} p_j(1-p_j), \tag{2.15}$$

$$\bar{a}(p) = \sum_{j=1}^{N} p_j(1-p_j)a_j \Big/ \sum_{j=1}^{N} p_j(1-p_j), \tag{2.16}$$

$$\tau^2(p) = N^{-1} \sum_{j=1}^{N} p_j(1-p_j)(a_j-\bar{a}(p))^2 = N^{-1} \sum_{j=1}^{N} p_j(1-p_j)a_j^2 - \sigma^2(p)\bar{a}^2(p), \tag{2.17}$$

$$\kappa_{3,i}(p) = N^{-1} \sum_{j=1}^{N} p_j(1-p_j)(1-2p_j)(a_j-\bar{a}(p))^i, \qquad i = 0,1,2,3, \tag{2.18}$$

$$\kappa_{4,i}(p) = N^{-1} \sum_{j=1}^{N} p_j(1-p_j)(1-6p_j+6p_j^2)(a_j-\bar{a}(p))^i, \qquad i = 0,1,\ldots,4. \tag{2.19}$$

In this notation we shall suppress the dependence on p when this is convenient. Let ℓ denote Lebesgue measure on R^1 and define

$$\gamma(\varepsilon,\zeta,p) = \ell\{x: \exists_j\ |x-a_j| < \zeta,\ \varepsilon \le p_j \le 1-\varepsilon\}. \tag{2.20}$$

LEMMA 2.2. *Suppose that positive numbers* c, C, δ *and* ε *exist such that*

$$\tau^2(p) \ge c, \qquad \frac{1}{N} \sum_{j=1}^{N} a_j^4 \le C, \tag{2.21}$$

$$\gamma(\varepsilon,\zeta,p) \ge \delta N\zeta \quad \textit{for some } \zeta \ge N^{-3/2}\log N. \tag{2.22}$$

Then there exist positive numbers b, B *and* β *depending only on* c, C, δ *and* ε *such that*

$$|\psi(s,t,p)| \le BN^{-\beta\log N} \tag{2.23}$$

for all pairs (s,t) *such that* $|s| \le \pi N^{\frac{1}{2}}$, $|t| \le bN^{3/2}$ *and either* $|s| \ge \log(N+1)$ *or* $|t| \ge \log(N+1)$.

PROOF.

$$(2.24)\qquad |\psi(s,t,p)| = \prod_{j=1}^{N} [1 - 2p_j(1-p_j)\{1 - \cos(N^{-\frac{1}{2}}(s+a_jt))\}]^{\frac{1}{2}}$$

$$\le \exp\{-\sum_{j=1}^{N} p_j(1-p_j)[\tfrac{1}{2}N^{-1}(s+a_jt)^2 - \frac{1}{24}N^{-2}(s+a_jt)^4]\}$$

$$\le \exp\{-\tfrac{1}{2}[\tau^2t^2+\sigma^2(s+\bar{a}t)^2] + \frac{1}{12}N^{-1}[N^{-1}\sum_{j=1}^{N}(a_j-\bar{a})^4t^4 + (s+\bar{a}t)^4]\}.$$

Now (2.21) ensures that

$$(2.25)\qquad \sigma^2(p) \ge N\tau^4(p) \,/\, \sum_{j=1}^{N} a_j^4 \ge C^{-1}c^2,$$

$$(2.26)\qquad |\bar{a}(p)| \le [N^{-1}\sum_{j=1}^{N} a_j^4]^{\frac{1}{4}} \,/\, \sigma^2(p) \le c^{-2}C^{5/4},$$

and by (2.21), (2.24), (2.25) and (2.26) we conclude that there exist positive b_1, B and β depending only on c and C such that for $|s| \le b_1N^{\frac{1}{2}}$ and $|t| \le b_1N^{\frac{1}{2}}$

$$(2.27)\qquad |\psi(s,t,p)| \le B\exp\{-\beta(s^2+t^2)\}.$$

Next note that (2.25) and (2.21) imply that the number of indices j for which $p_j(1-p_j) \ge \frac{1}{2}c^2/C$ is at least $2Nc^2/C$ and the number of j for which $|a_j| \le (C/c)^{\frac{1}{2}}$ is at least $N - Nc^2/C$. Hence the number of indices j for which $|a_j| \le (C/c)^{\frac{1}{2}}$ and $p_j(1-p_j) \ge \frac{1}{2}c^2/C$ is at least Nc^2/C. Put $b_2 = \frac{1}{2}b_1(c/C)^{\frac{1}{2}}$ and we see that if $b_1N^{\frac{1}{2}} \le |s| \le \pi N^{\frac{1}{2}}$ and $|t| \le b_2N^{\frac{1}{2}}$, then for at least Nc^2/C indices j

$$[1 - 2p_j(1-p_j)\{1 - \cos(N^{-\frac{1}{2}}(s+a_jt))\}] \le 1 - c^2C^{-1}\{1 - \cos(\tfrac{b_1}{2})\}.$$

Combining this with (2.27) we see that it only remains to be shown that positive numbers b, B and β exist depending only on c, C, δ and ε and such that (2.23) holds for $|s| \le \pi N^{\frac{1}{2}}$ and $(b_1 \wedge b_2)N^{\frac{1}{2}} \le |t| \le bN^{3/2}$. For this we can appeal to the corresponding part of the proof of lemma 2.2 in ABZ (1976) with only minor modifications. □

Define functions $\mu_k(p)$, $1 \le k \le 6$, and $A_k(p)$, $0 \le k \le 6$, by

$$\mu_1 = \frac{\omega}{\sigma^2}, \quad \mu_2 = \frac{1}{\sigma^2} - \frac{\omega^2}{\sigma^4}, \quad \mu_3 = \frac{3\omega}{\sigma^4} - \frac{\omega^3}{\sigma^6},$$

$$\mu_4 = \frac{3}{\sigma^4} - \frac{6\omega^2}{\sigma^6} + \frac{\omega^4}{\sigma^8}, \quad \mu_5 = \frac{15\omega}{\sigma^6} - \frac{10\omega^3}{\sigma^8} + \frac{\omega^5}{\sigma^{10}},$$

$$\mu_6 = \frac{15}{\sigma^6} - \frac{45\omega^2}{\sigma^8} + \frac{15\omega^4}{\sigma^{10}} - \frac{\omega^6}{\sigma^{12}},$$

(2.29)
$$A_0 = 1 + \frac{N^{-\frac{1}{2}}}{6}\kappa_{3,0}\mu_3 + \frac{N^{-1}}{72}(3\kappa_{4,0}\mu_4 - \kappa_{3,0}^2\mu_6),$$

$$A_1 = -\frac{N^{-\frac{1}{2}}}{2}\kappa_{3,1}\mu_2 + \frac{N^{-1}}{12}(2\kappa_{4,1}\mu_3 - \kappa_{3,0}\kappa_{3,1}\mu_5),$$

$$A_2 = -\frac{N^{-\frac{1}{2}}}{2}\kappa_{3,2}\mu_1 + \frac{N^{-1}}{24}\{-6\kappa_{4,2}\mu_2 + (2\kappa_{3,0}\kappa_{3,2} + 3\kappa_{3,1}^2)\mu_4\},$$

$$A_3 = \frac{N^{-\frac{1}{2}}}{6}\kappa_{3,3} + \frac{N^{-1}}{36}\{-6\kappa_{4,3}\mu_1 + (\kappa_{3,0}\kappa_{3,3} + 9\kappa_{3,1}\kappa_{3,2})\mu_3\},$$

$$A_4 = \frac{N^{-1}}{24}\{\kappa_{4,4} - (2\kappa_{3,1}\kappa_{3,3} + 3\kappa_{3,2}^2)\mu_2\},$$

$$A_5 = -\frac{N^{-1}}{12}\kappa_{3,2}\kappa_{3,3}\mu_1,$$

$$A_6 = \frac{N^{-1}}{72}\kappa_{3,3}^2,$$

where we have suppressed the dependence on p. We shall show that

(2.30)
$$\tilde{\nu}(t,p) = \frac{(2\pi)^{\frac{1}{2}}}{\sigma(p)}\exp\{-\frac{\omega^2(p)}{2\sigma^2(p)} - \frac{\tau^2(p)t^2}{2} - i\omega(p)\bar{a}(p)t\}\sum_{k=0}^{6} A_k(p)(it)^k$$

is an asymptotic expansion for $\nu(t,p)$.

LEMMA 2.3. *Suppose that positive numbers* c, C, δ *and* ε *exist such that* (2.21) *and* (2.22) *are satisfied. Then there exist positive numbers* b, B *and* β *depending only on* c, C, δ *and* ε *such that for* $|t| \le bN^{3/2}$,

$$(2.31)\qquad |\nu(t,p) - \tilde{\nu}(t,p)| \le B[(N^{-3/2}+N^{-5/4}|t|^5)\exp\{-\frac{ct^2}{8}\} + N^{-\beta\log N}].$$

PROOF. In this proof b, b_i, B_i, β_i and N_0 denote appropriately chosen positive numbers depending only on c, C, δ and ε.

Arguing as in the proof of theorem 2.1 in ABZ (1976) we find by Taylor expansion of log ψ that if $|s + a_j t| \le \frac{1}{2}\pi N^{\frac{1}{2}}$ for all j, then

$$(2.32)\qquad \psi(s,t,p) = \exp\{i\omega s - \frac{\tau^2 t^2}{2} - \frac{\sigma^2(s+\bar{a}t)^2}{2}$$

$$- \frac{iN^{-3/2}}{6}\sum p_j(1-p_j)(1-2p_j)(s+a_jt)^3$$

$$+ \frac{N^{-2}}{24}\sum p_j(1-p_j)(1-6p_j+6p_j^2)(s+a_jt)^4 + M_1(s,t,p)\},$$

where

$$|M_1(s,t,p)| \le C_1 N^{-5/2}\sum |s+a_jt|^5$$

$$\le 16C_1(N^{-5/2}|t|^5\sum |a_j - \bar{a}|^5 + N^{-3/2}|s+\bar{a}t|^5)$$

for some absolute constant C_1. Now (2.21) and (2.26) imply that $N^{-1}\sum |a_j-\bar{a}|^3$, $N^{-1}\sum |a_j - \bar{a}|^4$, $N^{-1/4}\max|a_j|$ and $N^{-5/4}\sum |a_j - \bar{a}|^5$ are bounded. Using (2.21) and (2.25) we find that for all $|s| \le b_1 N^{\frac{1}{2}}$ and $|t| \le b_1 N^{\frac{1}{4}}$

$$\frac{N^{-3/2}}{6}\sum |s+a_jt|^3 + \frac{N^{-2}}{24}\sum (s+a_jt)^4 + |M_1(s,t,p)| \le \frac{\tau^2t^2 + \sigma^2(s+\bar{a}t)^2}{4}.$$

Hence further expansion of part of the exponential in (2.32) shows that

$$(2.33)\qquad \psi(s,t,p) = \tilde{\psi}(s,t,p) + M_2(s,t,p)$$

for $|s| \le b_1 N^{\frac{1}{2}}$ and $|t| \le b_1 N^{\frac{1}{4}}$, where

$$(2.34)\qquad \tilde{\psi}(s,t,p) = \exp\{i\omega s - \frac{\tau^2 t^2}{2} - \frac{\sigma^2(s+\bar{a}t)^2}{2}\}$$
$$\times [1 - \frac{iN^{-3/2}}{6} \sum p_j(1-p_j)(1-2p_j)(s+a_j t)^3$$
$$+ \frac{N^{-2}}{24} \sum p_j(1-p_j)(1-6p_j+6p_j^2)(s+a_j t)^4$$
$$- \frac{N^{-3}}{72} (\sum p_j(1-p_j)(1-2p_j)(s+a_j t)^3)^2] ,$$

$$(2.35)\qquad |M_2(s,t,p)| \le (N^{-3/2}+N^{-5/4}|t|^5) M_3(t,s+\bar{a}t)\exp\{-\frac{\tau^2 t^2 + \sigma^2(s+\bar{a}t)^2}{4}\}$$

and M_3 is a polynomial in t and $(s+\bar{a}t)$ of fixed degree with coefficients depending only on c and C. Therefore, for $|t| \le b_1 N^{\frac{1}{4}}$,

$$(2.36)\qquad \int_{-b_1 N^{\frac{1}{2}}}^{b_1 N^{\frac{1}{2}}} |\psi(s,t,p) - \tilde{\psi}(s,t,p)|ds \le B_1 (N^{-3/2}+N^{-5/4}|t|^5)\exp\{-\frac{ct^2}{8}\}.$$

Next we show that for $|t| \le b_1 N^{1/4}$,

$$(2.37)\qquad \int_{b_1 N^{\frac{1}{2}} \le |s| \le \pi N^{\frac{1}{2}}} |\psi(s,t,p)|ds \le B_2 N^{-\beta_2 \log N},$$

$$(2.38)\qquad \int_{|s| \ge b_1 N^{\frac{1}{2}}} |\tilde{\psi}(s,t,p)|ds \le B_3 N^{-\beta_3 \log N}.$$

For $N \ge N_0$, (2.37) is a consequence of lemma 2.2 and since $|\psi| \le 1$ we can choose B_2 so that (2.37) holds for all N. Because for all s and t

$$(2.39)\qquad |\tilde{\psi}(s,t,p)| \le \exp\{-\frac{\tau^2 t^2 + \sigma^2(s+\bar{a}t)^2}{2}\} M_4(t,s+\bar{a}t)$$

where M_4 is a polynomial depending only on c and C, (2.38) follows. Combining (2.11), (2.36), (2.37) and (2.38) we see that for $|t| \le b_1 N^{1/4}$

$$(2.40)\qquad |\nu(t,p) - \int_{-\infty}^{\infty} \tilde{\psi}(s,t,p)ds| \le B_4[(N^{-3/2}+N^{-5/4}|t|^5)\exp\{-\frac{ct^2}{8}\}$$
$$+ N^{-\beta_4 \log N}].$$

A direct application of lemma 2.2, the fact that $|\psi| \le 1$ and (2.39) show that we can choose B_4 and β_4 so that (2.40) continues to hold for $b_1 N^{1/4} \le |t| \le bN^{3/2}$ with b as in lemma 2.2.

It remains to be shown that for all s and t

$$\tilde{v}(t,p) = \int_{-\infty}^{\infty} \tilde{\psi}(s,t,p)ds. \tag{2.41}$$

This follows by straightforward but tedious computation using the fact that

$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left(\frac{z}{\sigma(p)} + \frac{i\omega(p)}{\sigma^2(p)}\right)^k e^{-\frac{1}{2}z^2} dz = \mu_k(p) \quad \text{for even } k$$
$$= i\mu_k(p) \quad \text{for odd } k. \ \square$$

We now turn to our asymptotic expansion for rejective sampling. For $1 \le k \le 6$, define functions $Q_k(p)$ by

$$Q_1 = -\frac{N^{-\frac{1}{2}}}{2}\kappa_{3,1}\mu_2 + \frac{N^{-1}}{6}[\kappa_{4,1}\mu_3 - 3\kappa_{3,0}\kappa_{3,1}(2\frac{\omega}{\sigma^6} - \frac{\omega^3}{\sigma^8})], \tag{2.42}$$

$$Q_2 = -\frac{N^{-\frac{1}{2}}}{2}\kappa_{3,2}\mu_1 + \frac{N^{-1}}{8}[-2\kappa_{4,2}\mu_2 + 2\kappa_{3,0}\kappa_{3,2}(\frac{1}{\sigma^4} - \frac{\omega^2}{\sigma^6}) + \kappa_{3,1}^2\mu_4],$$

$$Q_3 = \frac{N^{-\frac{1}{2}}}{6}\kappa_{3,3} + \frac{N^{-1}}{12}[-2\kappa_{4,3}\mu_1 + 3\kappa_{3,1}\kappa_{3,2}\mu_3],$$

$$Q_k = A_k, \qquad k = 4,5,6.$$

Let Φ and ϕ denote the standard normal d.f. and its density and let H_k denote the Hermite polynomial of degree k, thus

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \tag{2.43}$$

$$H_4(x) = x^4 - 6x^2 + 3, \quad H_5(x) = x^5 - 10x^3 + 15x.$$

We shall show that expansions for (2.8) and (2.9) are given by

$$\tilde{\rho}(t,p) = \exp\{-\frac{\tau^2(p)t^2}{2} - i\omega(p)\bar{a}(p)t\}[1 + \sum_{k=1}^{6} Q_k(p)(it)^k], \tag{2.44}$$

$$(2.45)\qquad \tilde{R}(x,p) = \Phi\left(\frac{x+\omega(p)\bar{a}(p)}{\tau(p)}\right) - \phi\left(\frac{x+\omega(p)\bar{a}(p)}{\tau(p)}\right)\sum_{k=1}^{6}\frac{Q_k(p)}{(\tau(p))^k}H_{k-1}\left(\frac{x+\omega(p)\bar{a}(p)}{\tau(p)}\right).$$

Note that $\tilde{\rho}$ is the Fourier-Stieltjes transform of $\tilde{R}$, i.e. $\tilde{\rho}(t,p) = \int e^{itx}d\tilde{R}(x,p)$.

THEOREM 2.1. *Suppose that positive numbers* c, C, D, δ *and* ε *exist such that* (2.21) *and* (2.22) *are satisfied and*

$$(2.46)\qquad |\omega(p)| \le D.$$

Then there exist positive numbers N_0 *and* B *depending only on* c, C, D, δ *and* ε *such that for* $N \ge N_0$, R(x,p) *is well-defined and*

$$(2.47)\qquad \sup_x |R(x,p) - \tilde{R}(x,p)| \le BN^{-5/4}.$$

PROOF. In this proof b, B_i, β, η and N_0 denote appropriately chosen positive numbers depending only on c, C, D, δ and ε.

By (2.21), (2.25), (2.26), (2.46) and lemma 2.3 we have for $N \ge N_0$,

$$(2.48)\qquad |\tilde{\nu}(0,p)| \ge \eta, \qquad |\nu(0,p) - \tilde{\nu}(0,p)| \le \frac{\eta}{2},$$

so that $|\nu(0,p)| \ge \eta/2 > 0$. In the first place it follows that for $N \ge N_0$, $c(p) > 0$ and hence (2.7) is satisfied and R(x,p) is properly defined. We assume that $N \ge N_0$ and we shall show that, with b as in lemma 2.3,

$$(2.49)\qquad \int_{-bN^{3/2}}^{bN^{3/2}} \left|\frac{\rho(t,p) - \tilde{\rho}(t,p)}{t}\right| dt \le B_1 N^{-5/4}.$$

By Esseen's smoothing lemma (Esseen (1945)) this suffices to prove the theorem because $\tilde{R}(-\infty,p) = 0$, $\tilde{R}(\infty,p) = 1$ and the derivative of $\tilde{R}$ with respect to x is bounded.

By (2.21), (2.25), (2.26) and (2.46), $\tilde{\rho}$ has a bounded derivative with respect to t. Also

$$\left|\frac{d\rho(t,p)}{dt}\right| \le N^{-\frac{1}{2}}\, E\Big(\Big|\sum a_j(W_j-p_j)\Big|\,\Big|\sum W_j=n\Big) \le N^{-\frac{1}{2}} \sum |a_j| \le C^{\frac{1}{4}}\, N^{\frac{1}{2}}.$$

Since $\rho(0,p) = \tilde{\rho}(0,t) = 1$, it follows that

$$(2.50)\qquad \int_{-N^{-2}}^{N^{-2}} \left|\frac{\rho(t,p) - \tilde{\rho}(t,p)}{t}\right| dt \le B_2 N^{-3/2}.$$

Next we note that (2.21), (2.25) and (2.26) ensure that for all t

$$(2.51)\qquad |\tilde{\nu}(t,p)| \le B_3 \exp\left\{-\frac{ct^2}{4}\right\}.$$

Together with (2.13), (2.48) and lemma 2.3 this implies that for $|t| \le bN^{3/2}$

$$(2.52)\qquad \left|\rho(t,p) - \frac{\tilde{\nu}(t,p)}{\tilde{\nu}(0,p)}\right| \le \frac{2}{\eta}\,|\nu(t,p) - \tilde{\nu}(t,p)| + \frac{2}{\eta^2}\,|\tilde{\nu}(t,p)| \cdot |\nu(0,p) - \tilde{\nu}(0,p)|$$

$$\le B_4\left[\left(N^{-3/2}+N^{-5/4}|t|^5\right)\exp\left\{-\frac{ct^2}{8}\right\} + N^{-\beta \log N}\right].$$

Again with the aid of (2.21), (2.25), (2.26) and (2.46) one can easily check that, for $1 \le k \le 6$, Q_k is obtained from A_k/A_0 by expanding the denominator and discarding all terms of order $N^{-3/2}$, i.e. that $|Q_k - A_k/A_0| \le B_5 N^{-3/2}$. It follows that

$$(2.53)\qquad |\tilde{\rho}(t,p) - \frac{\tilde{\nu}(t,p)}{\tilde{\nu}(0,p)}| \le B_6 N^{-3/2} \exp\left\{-\frac{ct^2}{4}\right\}$$

and combined with (2.52) this yields

$$(2.54)\qquad \int_{N^{-2}\le|t|\le bN^{3/2}} \left|\frac{\rho(t,p)-\tilde{\rho}(t,p)}{t}\right| dt \le B_7(N^{-3/2}\log N + N^{-5/4}) \le B_8 N^{-5/4}.$$

Together with (2.50) this proves (2.49) and the theorem. □

Two remarks should be made with regard to theorem 2.1. The first one concerns condition (2.46) that does not occur in the preceding lemmas. The meaning of this condition is perhaps obscured by the fact that we make it

do some odd jobs in the proof for which it is not really needed. We use it to show that (2.7) is satisfied for $N \geq N_0$, but (2.25) ensures that the number of indices j with $p_j = 0$ (or $p_j=1$) cannot exceed $m - C^{-1}c^2N + |\omega(p)|N^{\frac{1}{2}}$ (or $n - C^{-1}c^2N + |\omega(p)|N^{\frac{1}{2}}$) so that $|\omega(p)| \leq C^{-1}c^2N^{\frac{1}{2}}$ already implies (2.7) for all N. Condition (2.46) is also used to obtain (2.50), but in (2.50) we may replace N^{-2} by an arbitrarily high power of N^{-1} without doing any damage to the proof, and then the trivial bound $|\omega(p)| \leq N^{\frac{1}{2}}$ suffices. Finally we note that since

$$\min(\lambda,1-\lambda) \geq \sigma^2(p) - N^{-\frac{1}{2}}|\omega(p)|, \tag{2.55}$$

(2.46) forces λ to be bounded away from 0 and 1 for large N, which is obviously important although it does not show up explicitly in the proof. However, here $|\omega(p)| \leq \frac{1}{2}C^{-1}c^2N^{\frac{1}{2}}$ would be sufficient.

The basic function of assumption (2.46), however, is to avoid a large (or intermediate) deviation situation that the condition $\sum W_j = n$ would get us into if $\omega(p) = N^{-\frac{1}{2}}(E\sum W_j - n)$ would not be bounded. Technically speaking this is reflected in the proof at the point where (2.46) is used to show that $\nu(0,p)$ is bounded away from zero. Also (2.46) ensures that (2.45) provides an expansion in powers of $N^{-\frac{1}{2}}$ to the required order.

To see what happens when condition (2.46) is relaxed, we prefer not to try to adapt the proof of theorem 2.1 but to answer this question more directly by remarking that the conditional distribution of $\sum a_j W_j$ given $\sum W_j = n$ remains unchanged if we replace p by $\tilde{p}$ where $\tilde{p}_j/(1-\tilde{p}_j) = \xi p_j/(1-p_j)$ $1 \leq j \leq N$, for some $0 \leq \xi \leq \infty$. If (2.7) is satisfied there exists a unique ξ for which $\sum \tilde{p}_j = N\lambda$. Since $\omega(\tilde{p}) = 0$ it follows that if (2.21) and (2.22) are satisfied with p replaced by $\tilde{p}$, then (2.47) holds with $\tilde{R}(x,\tilde{p})$ instead of $\tilde{R}(x,p)$. Of course the snag is that in general $\tilde{p}$ can only be expressed analytically in terms of p as an infinite series. However, if $\omega(p) = O(N^{\alpha})$ for some $\alpha < \frac{1}{2}$, then a finite number of terms of this series will yield the required degree of accuracy and an explicit expansion for R(x,p) can be obtained. If $\alpha = 0$ this is expansion (2.45) but for $0 < \alpha < \frac{1}{2}$ more terms have to be included.

The second remark concerns the remainder $O(N^{-5/4})$ of our expansion. It is clear that by requiring that $\sum |a_j|^5 \leq CN$ in theorem 2.1 one obtains

$|R - \tilde{R}| \le BN^{-3/2} \log(N+1)$. Of course the "natural" order of the remainder is $O(N^{-3/2})$ and the factor $\log(N+1)$ is due only to technical difficulties in finding the conditional expectation of $\sum a_j W_j$ given $\sum W_j = n$.

The special case $p_j = \lambda$, $1 \le j \le N$, which is random sampling without replacement, is worth singling out because it corresponds to the null-hypothesis in the two-sample problem. Let $\bar{\lambda}$ denote the vector $(\lambda,\ldots,\lambda)$. For $p = \bar{\lambda}$, (2.45) simplifies to

$$\begin{aligned}(2.56)\qquad \tilde{R}(x,\bar{\lambda}) = \Phi\Big(\frac{x}{\tau(\bar{\lambda})}\Big) - \frac{\phi\Big(\frac{x}{\tau(\bar{\lambda})}\Big)}{\lambda(1-\lambda)}\Big[&\frac{\lambda(1-\lambda)}{2N} H_1\Big(\frac{x}{\tau(\bar{\lambda})}\Big) \\ &+ \frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}(1-2\lambda)}{6} \frac{\Sigma(a_j-a_\cdot)^3}{\{\Sigma(a_j-a_\cdot)^2\}^{3/2}} H_2\Big(\frac{x}{\tau(\bar{\lambda})}\Big) \\ &+ \Big\{\frac{1 - 6\lambda + 6\lambda^2}{24} \frac{\Sigma(a_j-a_\cdot)^4}{\{\Sigma(a_j-a_\cdot)^2\}^2} - \frac{(1-2\lambda)^2}{8N}\Big\} H_3\Big(\frac{x}{\tau(\bar{\lambda})}\Big) \\ &+ \frac{(1-2\lambda)^2}{72} \frac{\{\Sigma(a_j-a_\cdot)^3\}^2}{\{\Sigma(a_j-a_\cdot)^2\}^3} H_5\Big(\frac{x}{\tau(\bar{\lambda})}\Big)\Big],\end{aligned}$$

where

$$(2.57)\qquad \tau^2(\bar{\lambda}) = \frac{\lambda(1-\lambda)}{N} \sum_{j=1}^{N} (a_j-a_\cdot)^2,$$

$$(2.58)\qquad a_\cdot = \bar{a}(\bar{\lambda}) = \frac{1}{N} \sum_{j=1}^{N} a_j.$$

Define, with ℓ denoting Lebesgue measure on R^1,

$$(2.59)\qquad \gamma(\zeta) = \ell\{x: \exists_j |x - a_j| < \zeta\}.$$

For $p = \bar{\lambda}$, theorem 2.1 yields

COROLLARY 2.1. *Suppose that positive numbers* c, C, δ *and* ε *exist such that*

$$(2.60)\qquad \varepsilon \le \lambda \le 1 - \varepsilon,$$

$$(2.61)\qquad \frac{1}{N} \sum_{j=1}^{N} (a_j-a_\cdot)^2 \ge c, \qquad \frac{1}{N} \sum_{j=1}^{N} a_j^4 \le C,$$

(2.62) $\gamma(\zeta) \geq \delta N\zeta$ *for some* $\zeta \geq N^{-3/2} \log N$.

Then there exists $B > 0$ *depending only on* c, C, δ *and* ε *such that*

$$\sup_x |R(x,\bar{\lambda}) - \tilde{R}(x,\bar{\lambda})| \leq BN^{-5/4}.$$

Note that there is considerable further simplification in (2.56) if we either have almost equal sample sizes, i.e. $\lambda = \frac{1}{2} + O(N^{-3/4})$, or antisymmetric scores, i.e. $a_j + a_{N-j+1}$ is constant for all j. The latter happens for the locally most powerful rank test against shift alternatives when the underlying distribution is symmetric. In either case the H_2 and H_5 terms disappear so that the correction to the leading normal term is of order N^{-1} only and is due solely to a correction to the variance, the H_1 term, and a kurtosis correction corresponding to H_3.

3. AN UNCONDITIONAL EXPANSION

We encounter several difficulties on the way to a usable unconditional expansion.

(i) The distribution of Z is awkward to handle analytically;

(ii) As in ABZ (1976), the random variables obtained by substituting P for p in $\tilde{\rho}$ or $\tilde{R}$ are generally not summable;

(iii) Again as in ABZ (1976), final simplification is not possible with our present techniques unless we assume that the sequence of alternatives is contiguous to the hypothesis as $N \to \infty$.

In this section we shall deal with the first two difficulties. Although we do not assume contiguity we shall be governed in the form of our expansion, which will involve polynomials in $(P_j-\lambda)$, in the number of terms that we calculate and in what we relegate to the remainder by the consideration that we expect $P_j = \lambda + O_P(N^{-\frac{1}{2}})$ and $\sum (P_j-\lambda) = O_P(1)$.

Recall that we assumed that $X_1,\ldots,X_N$ are independent, $X_1,\ldots,X_m$ having common density f and $X_{m+1},\ldots,X_N$ having density g. We shall write P for probabilities and E for expectations calculated under this model. In addition we need to consider an auxiliary model where $X_1,\ldots,X_N$ are i.i.d. with common density $h = (1-\lambda)f + \lambda g$ and d.f. $H = (1-\lambda)F + \lambda G$. We shall write P_H for probabilities, E_H for expectations and σ_H^2 for variances calculated under this second model.

To simplify our notation we assume from this point on that

$$\sum_{j=1}^{N} a_j = 0. \tag{3.1}$$

Since $T = \sum (a_j-a_\cdot)V_j + na_\cdot$ it is obvious how all expansions need to be modified if (3.1) does not hold.

We meet difficulty (i) through

LEMMA 3.1.

$$E \exp\{itN^{-\frac{1}{2}}T\} = \frac{E_H \nu(t,P)\exp\{itN^{-\frac{1}{2}}\sum a_j P_j\}}{2\pi N^{\frac{1}{2}} B_{N,n}(\lambda)}, \tag{3.2}$$

where

$$B_{N,n}(\lambda) = \binom{N}{n}\lambda^n(1-\lambda)^{N-n}.$$

PROOF. Under our original model the density of Z at the point $z = (z_1,\ldots,z_N)$ with $z_1 < z_2 < \ldots < z_N$ is given by

$$\sum \prod_{j=1}^{m} f(z_{i_j}) \prod_{j=m+1}^{N} g(z_{i_j}),$$

where the sum ranges over all permutations $i_1,\ldots,i_N$ of $1,\ldots,N$. Under our second model this density is

$$N! \prod_{j=1}^{N} [(1-\lambda)f(z_j) + \lambda g(z_j)].$$

By the Radon-Nikodym theorem and lemma 2.1,

$$E \exp\{itN^{-\frac{1}{2}}T\} = E \frac{\nu(t,P)}{\nu(0,P)} \exp\{itN^{-\frac{1}{2}} \sum a_j P_j\}$$

$$= E_H \frac{\nu(t,P)}{\nu(0,P)} \exp\{itN^{-\frac{1}{2}} \sum a_j P_j\} \sum \prod_{j=1}^{m} \frac{f(Z_{i_j})}{h(Z_{i_j})} \prod_{j=m+1}^{N} \frac{g(Z_{i_j})}{h(Z_{i_j})} \frac{1}{N!}$$

$$= [B_{N,n}(\lambda)]^{-1} E_H \frac{\nu(t,P)}{\nu(0,P)} \exp\{itN^{-\frac{1}{2}} \sum a_j P_j\} c(P),$$

where c is defined by (2.6) or (2.12). The lemma follows from (2.11) and (2.13). □

Lemma 3.1 shows that we are concerned with $\tilde{\nu}$ rather than $\tilde{\rho}$, but since $\tilde{\nu}$ as a function of P is no more summable than $\tilde{\rho}$, we still have to face difficulty (ii). We do this by showing that $\tilde{\nu}$ may be replaced by a summable function ν^* outside a set that will later be seen to have sufficiently small probability. Define

$$(3.3) \qquad \nu^*(t,p) = \left[\frac{2\pi}{\lambda(1-\lambda)}\right]^{\frac{1}{2}} \exp\left\{-\frac{\lambda(1-\lambda)}{2N} \sum a_j^2 t^2\right\} \sum_{k=0}^{6} A_k^*(p)(it)^k,$$

where

$$(3.4)\qquad A_0^*(p) = 1 + \frac{1}{2\lambda(1-\lambda)N}\left[\sum (p_j-\lambda)^2 - \left\{\sum (p_j-\lambda)\right\}^2 - \frac{1-\lambda+\lambda^2}{6}\right],$$

$$A_1^*(p) = N^{-3/2}\sum a_j p_j\left[1 - \frac{1-2\lambda}{\lambda(1-\lambda)}\sum (p_j-\lambda)\right],$$

$$A_2^*(p) = \frac{(1-2\lambda)}{2N}\sum a_j^2(p_j-\lambda) - \frac{\Sigma a_j^2}{2N^2}\left[(1-2\lambda)\sum(p_j-\lambda) - \lambda(1-\lambda)\right]$$

$$- \frac{1}{2N}\sum a_j^2(p_j-\lambda)^2 - \frac{(1-2\lambda)^2}{2\lambda(1-\lambda)N^2}\left\{\sum a_j p_j\right\}^2,$$

$$A_3^*(p) = \frac{N^{-3/2}}{6}\left[\lambda(1-\lambda)(1-2\lambda)\sum a_j^3 + (1-6\lambda+6\lambda^2)\sum a_j^3(p_j-\lambda)\right.$$

$$\left. - \frac{3}{N}(1-2\lambda)^2\sum a_j^2\sum a_j p_j\right],$$

$$A_4^*(p) = \frac{\lambda(1-\lambda)(1-6\lambda+6\lambda^2)}{24N^2}\sum a_j^4 - \frac{\lambda(1-\lambda)(1-2\lambda)^2}{8N^3}\left\{\sum a_j^2\right\}^2$$

$$+ \frac{(1-2\lambda)^2}{8N^2}\left\{\sum a_j^2(p_j-\lambda)\right\}^2,$$

$$A_5^*(p) = \frac{N^{-5/2}}{12}\lambda(1-\lambda)(1-2\lambda)^2\sum a_j^3\sum a_j^2(p_j-\lambda),$$

$$A_6^*(p) = \frac{\lambda^2(1-\lambda)^2(1-2\lambda)^2}{72N^3}\left\{\sum a_j^3\right\}^2.$$

LEMMA 3.2. *Suppose that* (3.1) *holds and that positive numbers* c, C *and* ε *exist such that* (2.21) *is satisfied and*

$$(3.5)\qquad \varepsilon \le \lambda \le 1 - \varepsilon.$$

Then there exist positive numbers B *and* β *depending only on* c, C *and* ε *such that*

$$(3.6)\qquad |\tilde{\nu}(t,p) - \nu^*(t,p)| \le B\exp\{-\beta t^2\}\left[\{N^{-3/2} + N^{-5/4}|t|\}\{1 + N\sum(p_j-\lambda)^4\}\right.$$

$$\left. + N^{-3/2}\left\{\sum (p_j-\lambda)^4\right\}\right].$$

PROOF. For simplicity we make use of order symbols in this proof and $O(x)$ will denote a quantity that is bounded by $B_1|x|$ where B_1 depends only on c, C and ε.

Suppose first that $|\omega(p)| > 1$. Then (2.21) and (3.5) are easily seen to imply that $|\nu^*(t,p)| = O(\omega^2(p)\exp\{-\varepsilon(1-\varepsilon)ct^2/4\})$, whereas for $\tilde{\nu}(t,p)$ we have the bound (2.51). The right-hand side of (3.6), however, contains a term $BN^{\frac{1}{2}}\omega^4(p)\exp\{-\beta t^2\}$ so that the lemma is trivial for $|\omega(p)| > 1$.

We therefore assume that $|\omega(p)| \le 1$. Noting that $\sigma^2(p)$ is bounded away from zero (c.f. (2.25)), we expand σ^{-2}, $\bar{a}$, τ^2 and $\kappa_{r,i}$ about the point $p_j = \lambda$, $1 \le j \le N$, using elementary inequalities to bound the remainders in terms of N and

$$M_1 = N^{-1} \sum (p_j-\lambda)^4, \qquad M_2 = N^{-1} \left|\sum (p_j-\lambda)\right|.$$

We find

$$\frac{1}{\sigma^2(p)} = \frac{1}{\lambda(1-\lambda)}\left[1 - \frac{(1-2\lambda)}{\lambda(1-\lambda)N}\sum (p_j-\lambda) + \frac{1}{\lambda(1-\lambda)N}\sum (p_j-\lambda)^2\right]$$

$$+ O(M_1+M_2^2) = \frac{1}{\lambda(1-\lambda)} + O(M_1^{\frac{1}{2}}+M_2) ,$$

$$\bar{a}(p) = \frac{(1-2\lambda)}{\lambda(1-\lambda)N}\sum a_j p_j + O(M_1^{\frac{1}{2}}) = O(M_1^{\frac{1}{4}}) ,$$

$$\tau^2(p) = \frac{\lambda(1-\lambda)}{N}\sum a_j^2 + \frac{(1-2\lambda)}{N}\sum a_j^2(p_j-\lambda) - \frac{1}{N}\sum a_j^2(p_j-\lambda)^2$$

$$- \frac{(1-2\lambda)^2}{\lambda(1-\lambda)N^2}\left\{\sum a_j p_j\right\}^2 + O(M_1^{3/4}) ,$$

$$\kappa_{3,0}(p) = \lambda(1-\lambda)(1-2\lambda) + O(M_1^{\frac{1}{2}}+M_2) ,$$

$$\kappa_{3,1}(p) = -\frac{2\lambda(1-\lambda)}{N}\sum a_j p_j + O(M_1^{\frac{1}{2}}) ,$$

$$\kappa_{3,2}(p) = \frac{\lambda(1-\lambda)(1-2\lambda)}{N}\sum a_j^2 + O(M_1^{\frac{1}{4}}) ,$$

$$\kappa_{3,3}(p) = \frac{\lambda(1-\lambda)(1-2\lambda)}{N}\sum a_j^3 + \frac{(1-6\lambda+6\lambda^2)}{N}\sum a_j^3(p_j-\lambda)$$

$$- \frac{3(1-2\lambda)^2}{N^2}\sum a_j^2 \sum a_j p_j + O(N^{\frac{1}{4}}M_1^{\frac{1}{2}})(=O(1)) ,$$

$$\kappa_{4,0}(p) = \lambda(1-\lambda)(1-6\lambda+6\lambda^2) + O(M_1^{\frac{1}{4}}) ,$$

$$\kappa_{4,1}(p) = O(M_1^{\frac{1}{4}}) ,$$

$$\kappa_{4,2}(p) = \frac{\lambda(1-\lambda)(1-6\lambda+6\lambda^2)}{N}\sum a_j^2 + O(M_1^{\frac{1}{4}}) ,$$

$$\kappa_{4,3}(p) = O(1) ,$$

$$\kappa_{4,4}(p) = \frac{\lambda(1-\lambda)(1-6\lambda+6\lambda^2)}{N}\sum a_j^4 + O(N^{\frac{1}{4}}M_1^{\frac{1}{4}})(=O(1)).$$

To illustrate the computations involved we present the argument for $\kappa_{3,3}$. By (2.21), the result for $\bar{a}(p)$ and the fact that $0 \le M_1 \le 1$, we have

$$\kappa_{3,3}(p) = N^{-1}\sum p_j(1-p_j)(1-2p_j)a_j^3 - 3N^{-1}\bar{a}(p)\sum p_j(1-p_j)(1-2p_j)a_j^2$$

$$+ O(M_1^{\frac{1}{2}}) = N^{-1}\lambda(1-\lambda)(1-2\lambda)\sum a_j^3 + N^{-1}(1-6\lambda+6\lambda^2)\sum a_j^3(p_j-\lambda)$$

$$- 3N^{-2}(1-2\lambda)^2\sum a_j^2\sum a_j p_j + O(M_1^{\frac{1}{2}}+N^{-1}\sum|a_j|^3(p_j-\lambda)^2$$

$$+ N^{-1}M_1^{\frac{1}{4}}\sum a_j^2|p_j-\lambda|).$$

Hölder's inequality and (2.21) imply that

$$N^{-1}M_1^{1/4}\sum a_j^2|p_j-\lambda| \le N^{-1}M_1^{1/4}\left(\sum|a_j|^{8/3}\right)^{3/4}(NM_1)^{1/4} = O(M_1^{1/2}) ,$$

$$N^{-1}\sum|a_j|^3(p_j-\lambda)^2 \le N^{-1}(CN)^{1/4}\sum a_j^2(p_j-\lambda)^2 \le C^{1/4}N^{-3/4}(NM_1\sum a_j^4)^{1/2}$$

$$= O(N^{1/4}M_1^{1/2}).$$

As $\bar{a}(p)$ is bounded, $\kappa_{3,3}(p)$ is obviously also $O(1)$. Note that the a-typical order of the remainder $O(N^{\frac{1}{4}}M_1^{\frac{1}{2}})$ originates from the term $O(N^{-1}\sum|a_j|^3(p_j-\lambda)^2)$ where we have to sacrifice a factor $O(N^{-\frac{1}{4}})$ in order to apply Hölder's inequality and (2.21). The same thing occurs for $\kappa_{4,4}(p)$.

For $\mu_k(p)$ defined by (2.28) we find

$$\mu_1(p) = \frac{1}{\lambda(1-\lambda)N^{\frac{1}{2}}}\sum(p_j-\lambda) + O(M_1^{\frac{1}{2}}+N^{\frac{1}{2}}M_2^2) ,$$

$$\mu_2(p) = \frac{1}{\lambda(1-\lambda)} + O(M_1^{\frac{1}{2}}+M_2+NM_2^2) ,$$

$$\mu_3(p) = \frac{3}{\lambda^2(1-\lambda)^2N^{\frac{1}{2}}}\sum(p_j-\lambda) + O(M_1^{\frac{1}{2}}+NM_2^2) ,$$

$$\mu_4(p) = \frac{3}{\lambda^2(1-\lambda)^2} + O(M_1^{\frac{1}{2}}+M_2+NM_2^2) ,$$

$$\mu_5(p) = O(N^{\frac{1}{2}}M_2) ,$$

$$\mu_6(p) = \frac{15}{\lambda^3(1-\lambda)^3} + O(M_1^{\frac{1}{2}}+M_2+NM_2^2) .$$

Straightforward but tedious calculation now yields

$$\begin{aligned}(3.7)\qquad \sum_{k=0}^{6} A_k(p)(it)^k = {} & \left[1 + \frac{(1-2\lambda)}{2\lambda(1-\lambda)N}\sum(p_j-\lambda) - \frac{(1-\lambda+\lambda^2)}{12\lambda(1-\lambda)N}\right] \\ & + \frac{\sum a_j p_j}{N^{3/2}}\,it - \frac{\sum a_j^2}{2N^2}\left[(1-2\lambda)\sum(p_j-\lambda) - \lambda(1-\lambda)\right](it)^2 \\ & + \frac{1}{6N^{3/2}}\left[\lambda(1-\lambda)(1-2\lambda)\sum a_j^3 + (1-6\lambda+6\lambda^2)\sum a_j^3(p_j-\lambda) - \frac{3(1-2\lambda)^2}{N}\sum a_j^2\sum a_j p_j\right](it)^3 \\ & + \frac{\lambda(1-\lambda)}{24N^2}\left[(1-6\lambda+6\lambda^2)\sum a_j^4 - \frac{3(1-2\lambda)^2}{N}\left\{\sum a_j^2\right\}^2\right](it)^4 \\ & + \frac{\lambda^2(1-\lambda)^2(1-2\lambda)^2}{72N^3}\left\{\sum a_j^3\right\}^2(it)^6 + O((|t|^3+t^4)[N^{-5/4} + N^{-1/4}M_1^{1/2}] \\ & + (1+t^6)[N^{-3/2} + N^{-1/2}M_1^{1/2} + N^{1/2}M_2^2]) .\end{aligned}$$

Next we expand the remaining factor in (2.30). Because both $\tau^2(p)$ and its leading term $\lambda(1-\lambda)N^{-1}\sum a_j^2$ are bounded away from zero, there exists $\beta > 0$ depending only on c, C and D, such that

$$\frac{(2\pi)^{\frac{1}{2}}}{\sigma(p)} \exp\left\{-\frac{\omega^2(p)}{2\sigma^2(p)} - \frac{\tau^2(p)t^2}{2} - i\omega(p)\bar{a}(p)t\right\}$$

$$= \left[\frac{2\pi}{\lambda(1-\lambda)}\right]^{\frac{1}{2}} \exp\left\{-\frac{\lambda(1-\lambda)}{2N}\sum a_j^2 t^2\right\}$$

$$\times \left[1 - \frac{1}{2\lambda(1-\lambda)N}\left\{(1-2\lambda)\sum(p_j-\lambda) - \sum(p_j-\lambda)^2 + \{\sum(p_j-\lambda)\}^2\right\}\right.$$

$$- \frac{(1-2\lambda)}{\lambda(1-\lambda)N^{3/2}}\sum(p_j-\lambda)\sum a_j p_j(it) + \frac{1}{2N}\left\{(1-2\lambda)\sum a_j^2(p_j-\lambda)\right.$$

$$\left.\left. - \sum a_j^2(p_j-\lambda)^2 - \frac{(1-2\lambda)^2}{\lambda(1-\lambda)N}\{\sum a_j p_j\}^2\right\}(it)^2\right.$$

$$\left. + \frac{(1-2\lambda)^2}{8N^2}\{\sum a_j^2(p_j-\lambda)\}^2(it)^4\right]$$

$$+ O\left(\exp\{-\beta t^2\}\left[N^{-3/2} + N^{1/2}M_1 + N^{5/2}M_2^4\right]\right) .$$

Multiplication by (3.7) yields (3.6). □

Here is our first unconditional expansion. Define

$$\rho(t) = E\exp\{itN^{-\frac{1}{2}}T\} , \tag{3.8}$$

$$\rho^*(t) = \exp\left\{-\frac{\lambda(1-\lambda)}{2N}\sum a_j^2 t^2\right\}E_H\left[\exp\{itN^{-\frac{1}{2}}\sum a_j P_j\}\right. \tag{3.9}$$

$$\left.\times\left\{1 + \frac{1}{2\lambda(1-\lambda)N}\left(\sum(P_j-\lambda)^2 - \{\sum(P_j-\lambda)\}^2\right) + \sum_{k=1}^{6} A_k^*(P)(it)^k\right\}\right] .$$

LEMMA 3.3. *Suppose that* (3.1) *holds and that positive numbers* c, C, δ, δ' *and* ε *exist with* $\delta' < \min(\frac{1}{2}, \delta/2, c^2C^{-1}/4)$ *and such that* (2.62) *is satisfied and*

(3.10) $\quad \frac{1}{N}\sum a_j^2 \geq c, \qquad \frac{1}{N}\sum a_j^4 \leq C,$

(3.11) $\quad P_H\left(\varepsilon \leq \frac{\lambda g(X_1)}{h(X_1)} \leq 1-\varepsilon\right) \geq 1 - \delta'.$

Then there exist positive numbers b, B, β_1 *and* β_2 *depending only on* c, C, δ, δ' *and* ε *such that for* $|t| \leq bN^{3/2}$,

(3.12) $$|\rho(t) - \rho^*(t)| \leq B\left[\exp\{-\beta_1 t^2\}(N^{-3/2}+N^{-5/4}|t|)\left\{1 + N^2 E_H\left(\frac{g(X_1)}{h(X_1)} - 1\right)^4\right\} + N^{-\beta_2 \log N}\right].$$

PROOF. In this proof we again use O symbols that are uniform for fixed c, C, δ, δ' and ε. Note that $E_H\{g(X_1)/h(X_1)\} = 1$, so that (3.11) and Markov's inequality ensure that $\min(\lambda, 1-\lambda) \geq \varepsilon(1-\delta')$.

Take a number $\delta'' \in (\delta', \min(1/2, \delta/2, c^2C^{-1}/4)$ and define the event E by

$$E = \{\varepsilon \leq P_j \leq 1 - \varepsilon \text{ for at least } (1-\delta'')N \text{ indices } j\} =$$

$$= \{\varepsilon \leq \frac{\lambda g(X_j)}{h(X_j)} \leq 1 - \varepsilon \text{ for at least } (1-\delta'')N \text{ indices } j\}.$$

Applying an exponential bound for binomial probabilities (c.f. Okamoto (1958)) we find that (3.11) implies

(3.13) $\quad P_H(E) \geq 1 - \exp\{-2N(\delta''-\delta')^2\}.$

Because λ and $(1-\lambda)$ are bounded away from 0, the same is true for $N^{\frac{1}{2}} B_{N,n}(\lambda)$. Also, (2.10) and (2.11) imply that $|\nu(t,p)| \leq 2\pi N^{\frac{1}{2}}$ for all t and p. Hence application of lemma 3.1 shows that

(3.14) $$\rho(t) = \frac{E_H \nu(t,P)\exp\{itN^{-\frac{1}{2}}\sum a_j P_j\}\chi_E}{2\pi N^{\frac{1}{2}} B_{N,n}(\lambda)} + O(\exp\{-N(\delta''-\delta')^2\}),$$

where χ_E denotes the indicator of E.

Since $\delta'' < \delta/2$, (2.62) ensures the validity of (2.22) on the set E with δ replaced by $\delta - 2\delta''$. If $\sum'$ denotes summation over those indices j

for which $P_j \notin [\varepsilon,1-\varepsilon]$ and k denotes the number of these indices, then $k \le \delta''N$ on E and as a result

$$\tau^2(P) \ge \frac{\varepsilon(1-\varepsilon)}{N}\left[\sum_{j=1}^{N}(a_j-\bar{a}(P))^2 - \sum{}'(a_j-\bar{a}(P))^2\right]$$

$$\ge \frac{\varepsilon(1-\varepsilon)}{N}\left[\sum_{j=1}^{N} a_j^2 + N\{\bar{a}(P)\}^2 - 2\sum{}' a_j^2 - 2k\{\bar{a}(P)\}^2\right]$$

$$\ge \frac{\varepsilon(1-\varepsilon)}{N}\left[cN - 2\{k\sum{}' a_j^4\}^{\frac{1}{2}}\right] \ge \varepsilon(1-\varepsilon)[c - 2\{\delta''C\}^{\frac{1}{2}}] > 0$$

on E, because $\delta'' < \min(\frac{1}{2}, c^2C^{-1}/4)$.

We have shown that on the set E, a and P satisfy the conditions on a and p in lemmas 2.3 and 3.2. Combining (3.14), (2.31) and (3.6) we obtain

$$\rho(t) = \frac{E_H\nu^*(t,P)\exp\{itN^{-\frac{1}{2}}\sum a_jP_j\}\chi_E}{2\pi N^{\frac{1}{2}}B_{N,n}(\lambda)} + O(N^{-\beta_2\log N}) \tag{3.15}$$

$$+ \exp\{-\beta_1 t^2\}[\{N^{-3/2}+N^{-5/4}|t|\}\{1 + NE_H\sum(P_j-\lambda)^4\}$$

$$+ N^{-3/2}E_H\{\sum(P_j-\lambda)\}^4])$$

for $|t| \le bN^{3/2}$, where b, β_1 and β_2 depend on c, C, δ, δ' and ε only.

Because of (3.13) and the fact that $\nu^*(t,p) = O(N)$, (3.15) remains valid if we delete χ_E. Using

$$2\pi N^{\frac{1}{2}}B_{N,n}(\lambda) = \left[\frac{2\pi}{\lambda(1-\lambda)}\right]^{\frac{1}{2}}\left(1 - \frac{1-\lambda+\lambda^2}{12\lambda(1-\lambda)N}\right) + O(N^{-2})$$

one easily verifies that in (3.15) the first term on the right may be replaced by $\rho^*(t)$ without changing the order of the remainder. Since

$$E_H\sum(P_j-\lambda)^4 = E_H\sum\left(\frac{\lambda g(X_i)}{h(X_i)} - \lambda\right)^4 = \lambda^4 NE_H\left(\frac{g(X_1)}{h(X_1)} - 1\right)^4,$$

$$E_H\{\sum(P_j-\lambda)\}^4 = \lambda^4 E_H\left\{\sum\left(\frac{g(X_i)}{h(X_i)} - 1\right)\right\}^4 \le 3\lambda^4N^2E_H\left(\frac{g(X_1)}{h(X_1)} - 1\right)^4,$$

the proof of the lemma is complete. □

Define

$$\pi_j = E_H P_j, \qquad \pi = (\pi_1,\ldots,\pi_N) . \tag{3.16}$$

In the remaining part of this section we obtain a further expansion for $\rho(t)$ and convert this expansion into one for the d.f. of T. Although we still do not assume contiguity, we shall be guided in what terms we include in the remainder by the fact that under contiguous alternatives we expect $(P_j-\pi_j)$ to behave roughly like $O_{P_H}(N^{-1})$. Let

$$K(x) = \Phi(x) - \phi(x) \sum_{k=0}^{5} \alpha_k H_k(x) , \tag{3.17}$$

where Φ and ϕ denote the standard normal d.f. and its density, the Hermite polynomials H_k are given by (2.43) and

$$\alpha_0 = \frac{\sum a_j\pi_j}{\{\lambda(1-\lambda)\sum a_j^2\}^{\frac{1}{2}}N} , \tag{3.18}$$

$$\alpha_1 = \frac{\sigma_H^2(\Sigma a_jP_j)-\Sigma a_j^2E_H(P_j-\lambda)^2+(1-2\lambda)\Sigma a_j^2(\pi_j-\lambda)}{2\lambda(1-\lambda)\Sigma a_j^2} - \frac{(1-2\lambda)^2\{\Sigma a_j\pi_j\}^2}{2\lambda^2(1-\lambda)^2N\Sigma a_j^2} + \frac{1}{2N},$$

$$\alpha_2 = \frac{[\lambda(1-\lambda)(1-2\lambda)\Sigma a_j^3+(1-6\lambda+6\lambda^2)\Sigma a_j^3(\pi_j-\lambda)-3(1-2\lambda)^2N^{-1}\Sigma a_j^2\Sigma a_j\pi_j]}{6\{\lambda(1-\lambda)\Sigma a_j^2\}^{3/2}} ,$$

$$\alpha_3 = \frac{\lambda(1-\lambda)(1-6\lambda+6\lambda^2)\Sigma a_j^4-3\lambda(1-\lambda)(1-2\lambda)^2N^{-1}\{\Sigma a_j^2\}^2+3(1-2\lambda)^2\{\Sigma a_j^2(\pi_j-\lambda)\}^2}{24\{\lambda(1-\lambda)\Sigma a_j^2\}^2} ,$$

$$\alpha_4 = \frac{(1-2\lambda)^2\Sigma a_j^3\Sigma a_j^2(\pi_j-\lambda)}{12\{\lambda(1-\lambda)\}^{3/2}\{\Sigma a_j^2\}^{5/2}} ,$$

$$\alpha_5 = \frac{(1-2\lambda)^2\{\Sigma a_j^3\}^2}{72\lambda(1-\lambda)\{\Sigma a_j^2\}^3} .$$

THEOREM 3.1. *Suppose that* (3.1) *holds and that positive numbers* c, C, δ *and* ε *exist such that* (3.10) *and* (2.62) *are satisfied and*

(3.19) $\qquad \varepsilon \le \lambda \le 1 - \varepsilon \ .$

Then there exists $B > 0$ *depending only on* c, C, δ *and* ε *such that*

$$\text{(3.20)} \qquad \sup_x \left| P\left(\frac{T}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}} \le x\right) - K\left(x - \frac{\Sigma a_j \pi_j}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}}\right)\right|$$

$$\le B\left\{N^{-5/4} + N^{3/4} E_H\left(\frac{g(X_1)}{h(X_1)} - 1\right)^4 + N^{-\frac{1}{2}}\left[\sum\{E_H|P_j-\pi_j|^3\}^{4/9}\right]^{9/4}\right\} .$$

PROOF. In this proof B_i and β_i denote appropriately chosen positive numbers depending only on c, C, δ and ε. We shall have to consider the r.v.

$$\text{(3.21)} \qquad U = N^{-\frac{1}{2}} \sum a_j (P_j - \pi_j)$$

and we note that

$$\text{(3.22)} \qquad E_H|U|^3 \le N^{-3/2}\left[\sum|a_j|\{E_H|P_j-\pi_j|^3\}^{1/3}\right]^3$$

$$\le C^{3/4} N^{-3/4}\left[\sum\{E_H|P_j-\pi_j|^3\}^{4/9}\right]^{9/4} .$$

Since $\sup_x(1+|K(x)|) \le B_1(1+E_H U^2) \le B_1(2+E_H|U|^3)$ we may assume without loss of generality that $E_H|U|^3 \le 1$, because otherwise (3.20) is satisfied trivially for $B = 3B_1 C^{3/4}$. Hence $\sup_x(1+|K(x)|) \le 3B_1$ and similar bounds $|\alpha_k| \le B_2(1+E_H U^2) \le 3B_2$ and $\sup_x|K'(x)| \le 3B_3$ hold for $\alpha_0, \ldots, \alpha_5$ and for the derivative K' of K.

Take $\delta' = \min(1/4, \delta/4, c^2 C^{-1}/8)$. In view of $1 + |K| \le 3B_1$ it is again no loss of generality to assume that $E_H(g(X_1)/h(X_1)-1)^4 \le \delta'\varepsilon^4/16$, because otherwise (3.22) with $B = 48B_1/(\delta'\varepsilon^4)$ is trivially true. Hence by (3.19) and Markov's inequality

$$P_H\left(\tfrac{1}{2}\varepsilon \le \frac{\lambda g(X_1)}{h(X_1)} \le 1 - \tfrac{1}{2}\varepsilon\right) \ge P_H\left(\left|\frac{g(X_1)}{h(X_1)} - 1\right| \le \tfrac{1}{2}\varepsilon\right) \ge 1 - \delta' ,$$

so that the conditions of lemma 3.3 are satisfied and (3.12) holds.

The proof hinges on the expansion

$$\exp\{itN^{-\frac{1}{2}}\sum a_j P_j\} = \exp\{itN^{-\frac{1}{2}}\sum a_j \pi_j\}[1+itU+\tfrac{1}{2}(itU)^2] + O(|tU|^3)$$

and its truncation to fewer terms. We apply this expansion to (3.9) and in the resulting expression we replace P by π wherever this is possible without giving rise to remainder terms that would be awkward to handle at this point. Using elementary inequalities to seperate out and bound those parts of the remainder that depend on the $(P_j-\lambda)$ rather than on the $(P_j-\pi_j)$, we arrive at

$$(3.23)\qquad |\rho^*(t)-\bar{\rho}(t)| \le B_4|t|\exp\{-\beta_3 t^2\}\Big[N^{-3/2} + N^{1/2}E_H\Big(\frac{g(X_1)}{h(X_1)}-1\Big)^4 + E_H|U|^3 + N^{-1}E_H|U\sum a_j^2(P_j-\pi_j)| + N^{-2}E_H\{\sum a_j^2(P_j-\pi_j)\}^2\Big],$$

$$(3.24)\qquad \bar{\rho}(t) = \exp\Big\{itN^{-\frac{1}{2}}\sum a_j\pi_j - t^2\,\frac{\lambda(1-\lambda)}{2N}\sum a_j^2\Big\}\Big[1+\sum_{k=1}^{6}\alpha_{k-1}\Big(\frac{\lambda(1-\lambda)\Sigma a_j^2}{N}\Big)^{\frac{1}{2}k}(it)^k\Big].$$

Because $\max|a_j| \le (CN)^{\frac{1}{4}}$ we find by the same reasoning as in (3.22),

$$N^{-1}E_H|U\sum a_j^2(P_j-\pi_j)| + N^{-2}E_H\{\sum a_j^2(P_j-\pi_j)\}^2 \le B_5N^{-5/4}E_H\{\sum|a_j(P_j-\pi_j)|\}^2$$

$$\le B_5N^{-5/4}[1 + E_H\{\sum|a_j(P_j-\pi_j)|\}^3]$$

$$\le B_5N^{-5/4} + B_6N^{-1/2}\Big[\sum\{E_H|P_j-\pi_j|^3\}^{4/9}\Big]^{9/4}.$$

Together with (3.22) this shows that (3.23) may be reduced to

$$(3.25)\qquad |\rho^*(t)-\bar{\rho}(t)| \le B_7|t|\exp\{-\beta_3t^2\}\Big\{N^{-5/4} + N^{1/2}E_H\Big(\frac{g(X_1)}{h(X_1)}-1\Big)^4 + N^{-1/2}\Big[\sum\{E_H|P_j-\pi_j|^3\}^{4/9}\Big]^{9/4}\Big\}.$$

As $\alpha_0,\ldots,\alpha_5$ are bounded and $N^{-\frac{1}{2}}|\sum a_j\pi_j| \le C^{\frac{1}{4}}N^{\frac{1}{2}}$, we have $|\bar{\rho}'(t)| \le B_8N^{\frac{1}{2}}$ for all t. Since $|\rho'(t)| \le N^{-\frac{1}{2}}E|T| \le C^{\frac{1}{4}}N^{\frac{1}{2}}$ for all t and $\rho(0) = \bar{\rho}(0) = 1$,

(3.26) $$|\rho(t) - \bar{\rho}(t)| \le B_9 N^{\frac{1}{2}}|t| \quad \text{for all } t.$$

Combining lemma 3.3, (3.25) and (3.26) we find

(3.27) $$\int_{-bN^{3/2}}^{bN^{3/2}} \left|\frac{\rho(t)-\bar{\rho}(t)}{t}\right| dt \le B_9 N^{-3/2} + \int_{N^{-2}\le|t|\le bN^{3/2}} \left|\frac{\rho(t)-\bar{\rho}(t)}{t}\right| dt$$

$$\le B_{10}\left\{N^{-5/4}+N^{3/4}E_H\left(\frac{g(X_1)}{h(X_1)}-1\right)^4 + N^{-\frac{1}{2}}\left[\sum\{E_H|P_j-\pi_j|^3\}^{4/9}\right]^{9/4}\right\}.$$

Now $\bar{\rho}(t)$ is the Fourier-Stieltjes transform of $K(\{N^{\frac{1}{2}}x-\sum a_j\pi_j\}\{\lambda(1-\lambda)\sum a_j^2\}^{-\frac{1}{2}})$ as a function of x. This is a function of bounded variation assuming the values 0 and 1 at $-\infty$ and $+\infty$ and having a derivative that is bounded by $3B_3 c^{-\frac{1}{2}}\{\varepsilon(1-\varepsilon)\}^{-1}$ in absolute value. It follows from the smoothing lemma (ESSEEN (1945)) that

$$\sup_x \left|P(N^{-\frac{1}{2}}T\le x) - K\left(\frac{N^{\frac{1}{2}}x-\sum a_j\pi_j}{\{\lambda(1-\lambda)\sum a_j^2\}^{\frac{1}{2}}}\right)\right|$$

is bounded above by the right-hand side of (3.20). A change of scale completes the proof. □

Theorem 3.1 provides the basic expansion for the distribution of T under contiguous alternatives. Only first and second moments of functions of order statistics remain to be determined. In section 4 we shall be concerned with a further simplification of the expansion and a precise evaluation of the order of the remainder. With regard to this remainder we are in a seemingly less favorable position than we were at the same stage in the one-sample problem (c.f. ABZ (1976), theorem 2.3), because the third remainder term in (3.20) is larger than the corresponding term in the one-sample case by a factor $N^{\frac{1}{4}}$. This is due to the appearance of the remainder term $N^{-1}E_H|U\sum a_j^2(P_j-\pi_j)|$ that does not occur for the one-sample statistic. It will turn out, however, that we shall need only a slightly stronger condition than before to show that the remainder is still $O(N^{-5/4})$.

The conditions of theorem 3.1 concern only the sample ratio λ and the scores a. There are no assumptions about the underlying densities f and g

but this is merely a trick; obviously something like contiguity is needed to make the expansion meaningful in the sense that the remainder is at all small. With regard to the conditions on the scores, (3.10) acts as a safeguard against too rapid growth and (2.62) ensures that the a_j do not cluster too much around too few points, thus preventing a too pronounced lattice character of the distribution of T, as was pointed out in ABZ (1976). It was also noted there that in the important case of exact scores $a_j = EJ(U_{j:N})$, with $U_{1:N} < U_{2:N} < \ldots < U_{N:N}$ order statistics from the uniform distribution on (0,1), both (3.10) and (2.62) will be satisfied for all N with fixed c, C and δ if J is a continously differentiable, non-constant function on (0,1) with $\int J^4 < \infty$. The same is true for approximate scores $a_j = J(j/(N+1))$ provided that J is monotone near 0 and 1.

4. CONTIGUOUS LOCATION ALTERNATIVES

The analysis in this section will be carried out for contiguous location alternatives rather than for contiguous alternatives in general. The general case can be treated in much the same way as the location case, but the conditions as well as the results become more involved.

We recall some assumptions and notation from section 3 of ABZ (1976). Let F be a d.f. with a density f that is positive on R^1 and four times differentiable with derivatives $f^{(i)}$, $i = 1,\ldots,4$. Define

$$\psi_i = \frac{f^{(i)}}{f}, \qquad i = 1,\ldots,4, \tag{4.1}$$

and suppose that positive numbers ε' and C' exist such that for

$$m_1 = 6, \quad m_2 = 3, \quad m_3 = \frac{4}{3}, \quad m_4 = 1,$$
$$\sup\left\{ \int_{-\infty}^{\infty} |\psi_i(x+y)|^{m_i} f(x)dx : |y| \le \varepsilon' \right\} \le C', \qquad i = 1,\ldots,4. \tag{4.2}$$

So far, we have studied the distribution of T under the assumption that $X_1,\ldots,X_N$ are independent, $X_1,\ldots,X_m$ having common d.f. F and $X_{m+1},\ldots,X_N$ having d.f. G. We now add the assumptions that

$$G(x) = F(x-\theta) \tag{4.3}$$

for all x and that

$$0 \le \theta \le DN^{-\frac{1}{2}} \tag{4.4}$$

for some $D > 0$. Probabilities under this particular model will still be denoted by P. Note that (4.2), (4.3) and (4.4) together imply contiguity.

In section 3 we also introduced an auxiliary model where $X_1,\ldots,X_N$ are supposed to be i.i.d. with common d.f. $H = (1-\lambda)F + \lambda G$. In view of (4.3) this common d.f. now becomes $H(x) = (1-\lambda)F(x) + \lambda F(x-\theta)$. Probabilities, expectations and variances under this model will be denoted by P_H, E_H and σ_H^2 as before. Similarly, P_F, E_F and σ_F^2 will indicate probabilities,

expectations and variances under a third model where $X_1,\ldots,X_N$ are i.i.d. with common d.f. F. Note that for $\theta = 0$ these three models coincide.

Define

$$\tilde{K}(x) = \Phi(x) - \phi(x) \sum_{k=0}^{5} \tilde{\alpha}_k H_k(x) , \tag{4.5}$$

where

$$\begin{aligned}
\tilde{\alpha}_0 = \frac{1}{6}\left(\frac{\lambda(1-\lambda)}{\Sigma a_j^2}\right)^{\frac{1}{2}}\Big[& 3(1-2\lambda)\theta^2 \sum a_j E_F\psi_2(Z_j) - 6N^{-1}\theta \sum a_j E_F\psi_1(Z_j) \\
& - \theta^3\textstyle\sum a_j E_F\{(1-3\lambda+3\lambda^2)\psi_3(Z_j) - 6\lambda(1-\lambda)\psi_1(Z_j)\psi_2(Z_j) \\
& + 3\lambda(1-\lambda)\psi_1^3(Z_j)\}\Big] ,
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\tilde{\alpha}_1 = \frac{1}{8\Sigma a_j^2}\Big[& -4(1-2\lambda)\theta\textstyle\sum a_j^2 E_F\psi_1(Z_j) + 2(1-2\lambda)^2\theta^2\sum a_j^2 E_F\psi_2(Z_j) \\
& - 4\lambda(1-\lambda)\theta^2\textstyle\sum a_j^2 E_F\psi_1^2(Z_j) + 4\lambda(1-\lambda)\theta^2\sigma_F^2(\sum a_j\psi_1(Z_j)) \\
& - 4(1-2\lambda)^2N^{-1}\theta^2\{\textstyle\sum a_j E_F\psi_1(Z_j)\}^2 \\
& + \lambda(1-\lambda)(1-2\lambda)^2\theta^4\{\textstyle\sum a_j E_F\psi_2(Z_j)\}^2\Big] + \frac{1}{2N} ,
\end{aligned}$$

$$\begin{aligned}
\tilde{\alpha}_2 = \frac{1}{12\{\lambda(1-\lambda)\}^{\frac{1}{2}}(\Sigma a_j^2)^{3/2}}\Big[& 2(1-2\lambda)\textstyle\sum a_j^3 - 2(1-6\lambda+6\lambda^2)\theta\sum a_j^3 E_F\psi_1(Z_j) \\
& + 6(1-2\lambda)^2N^{-1}\theta\textstyle\sum a_j^2\sum a_j E_F\psi_1(Z_j) \\
& - 3\lambda(1-\lambda)(1-2\lambda)^2\theta^3\textstyle\sum a_j^2 E_F\psi_1(Z_j)\sum a_j E_F\psi_2(Z_j)\Big] ,
\end{aligned}$$

$$\begin{aligned}
\tilde{\alpha}_3 = \frac{1}{24\lambda(1-\lambda)(\Sigma a_j^2)^2}\Big[& (1-6\lambda+6\lambda^2)\textstyle\sum a_j^4 + 3\lambda(1-\lambda)(1-2\lambda)^2\theta^2\{\sum a_j^2 E_F\psi_1(Z_j)\}^2 \\
& + 2\lambda(1-\lambda)(1-2\lambda)^2\theta^2\textstyle\sum a_j^3\sum a_j E_F\psi_2(Z_j)\Big] - \frac{(1-2\lambda)^2}{8\lambda(1-\lambda)N} ,
\end{aligned}$$

$$\tilde{\alpha}_4 = -\frac{(1-2\lambda)^2\theta\Sigma a_j^3\Sigma a_j^2 E_F\psi_1(Z_j)}{12\{\lambda(1-\lambda)\}^{1/2}(\Sigma a_j^2)^{5/2}} ,$$

$$\tilde{\alpha}_5 = \frac{(1-2\lambda)^2(\Sigma a_j^3)^2}{72\lambda(1-\lambda)(\Sigma a_j^2)^3} ,$$

and let

$$\eta = -\left(\frac{\lambda(1-\lambda)}{\Sigma a_j^2}\right)^{\frac{1}{2}} \theta \sum a_j E_F\psi_1(Z_j). \tag{4.7}$$

We shall show that $\tilde{K}(x-\eta)$ is an expansion for the d.f. of $\{\lambda(1-\lambda)\sum a_j^2\}^{-\frac{1}{2}}T$. The expansion will be established in theorem 4.1 and an evaluation of the order of the remainder will be given in theorem 4.2.

Let $\pi(F,\theta)$ denote the power of the one-sided level α test based on T for the hypothesis $F = G$ against the alternative $G(x) = F(x-\theta)$. Suppose that

$$\varepsilon'' \le \alpha \le 1 - \varepsilon'' , \tag{4.8}$$

for some $\varepsilon'' > 0$. We shall prove that an expansion for $\pi(F,\theta)$ is given by

$$\tilde{\pi}(F,\theta) = 1 - \Phi(u_\alpha-\eta) + \phi(u_\alpha-\eta)\sum_{k=0}^{5}\tilde{\beta}_k H_k(u_\alpha-\eta) , \tag{4.9}$$

where $u_\alpha = \Phi^{-1}(1-\alpha)$ is the upper α-point of the standard normal distribution and

$$\tilde{\beta}_0 = \tilde{\alpha}_0 - \frac{(1-2\lambda)\Sigma a_j^3}{6\{\lambda(1-\lambda)\}^{1/2}(\Sigma a_j^2)^{3/2}}(u_\alpha^2-1) + 2\tilde{\alpha}_5(2u_\alpha^3-5u_\alpha) - \frac{u_\alpha}{2N} \tag{4.10}$$

$$- \left\{\frac{(1-6\lambda+6\lambda^2)\Sigma a_j^4}{24\lambda(1-\lambda)(\Sigma a_j^2)^2} - \frac{(1-2\lambda)^2}{8\lambda(1-\lambda)N}\right\}(u_\alpha^3-3u_\alpha) ,$$

$$\tilde{\beta}_1 = \tilde{\alpha}_1 + \tilde{\alpha}_5(u_\alpha^2-1)^2 - \frac{(1-2\lambda)^2}{12(\Sigma a_j^2)^2}\theta^2\sum a_j^3\sum a_j E_F\psi_2(Z_j)(u_\alpha^2-1) ,$$

$$\tilde{\beta}_2 = \tilde{\alpha}_2 - \tilde{\alpha}_4(u_\alpha^2-1) ,$$

$$\tilde{\beta}_3 = \tilde{\alpha}_3 - 2\tilde{\alpha}_5(u_\alpha^2-1) ,$$

$$\tilde{\beta}_k = \tilde{\alpha}_k \qquad \text{for } k = 4,5.$$

THEOREM 4.1. *Suppose that* (3.1) *and* (4.3) *hold and that positive numbers* c, C, C', D, δ, ε *and* ε' *exist such that* (3.10), (2.62), (3.19), (4.2) *and* (4.4) *are satisfied. Define*

$$(4.11)\qquad M = N^{-5/4} + N^{-1/2}\theta^3\Big[\sum\{E_F|\psi_1(Z_j) - E_F\psi_1(Z_j)|^3\}^{4/9}\Big]^{9/4}$$
$$+ N^{-3/4}\theta^3\Big[\sum\{E_F(\psi_2(Z_j) - E_F\psi_2(Z_j))^2\}^{2/3}\Big]^{3/2} .$$

Then there exists B > 0 *depending only on* c, C, C', D, δ, ε *and* ε' *such that*

$$(4.12)\qquad \sup_x\Big|P\Big(\frac{T}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}} \le x\Big) - \tilde{K}(x-\eta)\Big| \le BM .$$

If, in addition, (4.8) *is satisfied there exists* B' > 0 *depending only on* c, C, C', D, δ, ε, ε' *and* ε'' *such that*

$$(4.13)\qquad |\pi(F,\theta) - \tilde{\pi}(F,\theta)| \le B'M.$$

PROOF. The proof of (4.12) hinges on Taylor expansion with respect to θ of the moments under P_H of functions of $P = (P_1,\ldots,P_N)$ occurring in expansion (3.20). Since both H and P depend on θ the argument is highly technical and laborious and it is therefore given in the appendix. Theorem 3.1, corollary A.1, (A.12) and (A.13) immediately yield (4.12).

The one-sided level α test based on T rejects the hypothesis if $T\{\lambda(1-\lambda)\sum a_j^2\}^{-\frac{1}{2}} \ge \xi_\alpha$ with possible randomization if equality occurs. Using (4.12) for $\theta = 0$ (or corollary 2.1), (3.10), (3.19) and (4.8) we easily show that

$$(4.14)\qquad \xi_\alpha = u_\alpha + \frac{(1-2\lambda)\Sigma a_j^3}{6\{\lambda(1-\lambda)\}^{1/2}(\Sigma a_j^2)^{3/2}}(u_\alpha^2-1) - 2\tilde{\alpha}_5(2u_\alpha^3-5u_\alpha) + \frac{u_\alpha}{2N}$$
$$+ \Big\{\frac{(1-6\lambda+6\lambda^2)\Sigma a_j^4}{24\lambda(1-\lambda)(\Sigma a_j^2)^2} - \frac{(1-2\lambda)^2}{8\lambda(1-\lambda)N}\Big\}(u_\alpha^3-3u_\alpha) + O(N^{-5/4}) ,$$

where, in this proof, $O(x)$ denotes a quantity bounded by $B_1|x|$ with B_1 depending only on c, C, C', D, δ, ε, ε' and ε''. Because of (4.12),

$$\pi(F,\theta) = 1 - \tilde{K}(\xi_\alpha - \eta) + O(M) \ .$$

Using (4.14), (4.8) and the bounds provided by corollary A.1, we now expand $\tilde{K}(\xi_\alpha - \eta)$ about the point $(u_\alpha - \eta)$ and arrive at (4.13). □

Define

$$\Psi_i(t) = \psi_i(F^{-1}(t)) = \frac{f^{(i)}(F^{-1}(t))}{f(F^{-1}(t))} \ , \qquad i = 1,\ldots,4 \ . \tag{4.15}$$

THEOREM 4.2. *Let* M *be defined by* (4.11) *and suppose that positive numbers* D, C *and* δ *exist such that* (4.4) *is satisfied and that* $|\Psi_1'(t)| \le C\{t(1-t)\}^{-5/4+\delta}$ *and* $|\Psi_2'(t)| \le C\{t(1-t)\}^{-3/2+\delta}$. *Then there exist* $B > 0$ *depending only on* D, C *and* δ *such that*

$$M \le BN^{-5/4} .$$

PROOF. The proof is similar to that of corollary A2.1 in ABZ (1976). To deal with the second term of M we take $h = \Psi_1$ and replace 4/3 by 5/4 in the proof of that corollary. For the third term of M we take $h = \Psi_2$, replace 4/3 by 3/2, appeal to condition R_2 instead of R_3 and otherwise proceed as in the proof of corollary A2.1 of ABZ (1976). □

5. EXACT AND APPROXIMATE SCORES

A further simplification of the expansions in section 4 may be obtained if we make certain smoothness assumptions about the scores a_j. Consider a continuous function J on (0,1) and let $U_{1:N} < U_{2:N} < \ldots < U_{N:N}$ denote order statistics of a sample of size N from the uniform distribution on (0,1). For N = 1,2,... we define the exact scores generated by J by

$$(5.1) \qquad a_j = a_{j,N} = EJ(U_{j:N}), \qquad j = 1,\ldots,N,$$

and the approximate scores generated by J by

$$(5.2) \qquad a_j = a_{j,N} = J(\tfrac{j}{N+1}), \qquad j = 1,\ldots,N.$$

For exact scores and general J theorem 5.1 will provide expansions for the d.f. of T under contiguous location alternatives of type F and for the power of the rank test against these alternatives. In theorem 5.2 we consider the special case $J = -\Psi_1$, with Ψ_1 as in (4.15), for exact as well as approximate scores. Note that the exact scores generated by $-\Psi_1$ define the locally most powerful rank test.

As in section 4 of ABZ (1976) it is now no longer feasible to keep the order of the remainder in our expansions down to $O(N^{-5/4})$ and we shall be content with $o(N^{-1})$. Also as in ABZ (1976) we shall formulate the results in this section for a fixed scores generating function J and a fixed d.f. F, leaving the construction of uniformity classes to the reader.

DEFINITION 5.1. $\mathcal{J}$ is the class of functions J on (0,1) that are twice continuously differentiable and nonconstant on (0,1) and satisfy

$$(5.3) \qquad \int_0^1 J(t)dt = 0,$$

$$(5.4) \qquad \lim_{t\to 0,1} \{t(1-t)\}^{7/6} J'(t) = 0,$$

$$(5.5) \qquad \limsup_{t\to 0,1} t(1-t)\left|\frac{J''(t)}{J'(t)}\right| < \frac{3}{2}.$$

$\mathcal{F}$ is the class of d.f.'s F on R^1 with positive and four times differentiable densities f and such that, for $\psi_i = f^{(i)}/f$, $\Psi_i = \psi_i(F^{-1})$, $m_1 = 6$, $m_2 = 3$, $m_3 = \frac{4}{3}$, $m_4 = 1$,

$$(5.6) \qquad \limsup_{y\to 0} \int_{-\infty}^{\infty} |\psi_i(x+y)|^{m_i} f(x)dx < \infty, \qquad i = 1,\ldots,4,$$

$$(5.7) \qquad \limsup_{t\to 0,1} t(1-t)\left|\frac{\Psi_1''(t)}{\Psi_1'(t)}\right| < \frac{3}{2}.$$

Note that one can argue as in the proof of corollary A2.1 of ABZ (1976) to show that, in conjunction with (5.5), condition (5.4) is weaker than the assumption $\int J^6(t)dt < \infty$. Define

$$(5.8) \qquad \bar{\alpha}_0 = \frac{1}{6}\left(\frac{\lambda(1-\lambda)}{N\int J^2(t)dt}\right)^{\frac{1}{2}}\Big[3(1-2\lambda)N\theta^2\int J(t)\Psi_2(t)dt - 6\theta\int J(t)\Psi_1(t)dt$$

$$- N\theta^3\int J(t)\{(1-3\lambda+3\lambda^2)\Psi_3(t)-6\lambda(1-\lambda)\Psi_1(t)\Psi_2(t)+3\lambda(1-\lambda)\Psi_1^3(t)\}dt\Big],$$

$$\bar{\alpha}_1 = \frac{1}{8\int J^2(t)dt}\Big[-4(1-2\lambda)\theta\int J^2(t)\Psi_1(t)dt+2(1-2\lambda)^2\theta^2\int J^2(t)\Psi_2(t)dt$$

$$-4\lambda(1-\lambda)\theta^2\int J^2(t)\Psi_1^2(t)dt$$

$$+ 4\lambda(1-\lambda)\theta^2\iint J(s)J(t)\Psi_1'(s)\Psi_1'(t)[s\wedge t - st]dsdt$$

$$- 4(1-2\lambda)^2\theta^2\{\int J(t)\Psi_1(t)dt\}^2$$

$$+ \lambda(1-\lambda)(1-2\lambda)^2N\theta^4\{\int J(t)\Psi_2(t)dt\}^2\Big] + \frac{1}{2N},$$

$$\bar{\alpha}_2 = \frac{1}{12\{\lambda(1-\lambda)N\}^{1/2}\{\int J^2(t)dt\}^{3/2}}\Big[2(1-2\lambda)\int J^3(t)dt$$

$$- 2(1-6\lambda+6\lambda^2)\theta\int J^3(t)\Psi_1(t)dt +$$

$$+ 6(1-2\lambda)^2\theta\int J^2(t)dt\int J(t)\Psi_1(t)dt$$

$$- 3\lambda(1-\lambda)(1-2\lambda)^2N\theta^3\left[\int J^2(t)\Psi_1(t)dt\int J(t)\Psi_2(t)dt\right] ,$$

$$\bar{\alpha}_3 = \frac{1}{24\lambda(1-\lambda)N\{\int J^2(t)dt\}^2}\left[(1-6\lambda+6\lambda^2)\int J^4(t)dt\right.$$

$$+ 3\lambda(1-\lambda)(1-2\lambda)^2N\theta^2\{\int J^2(t)\Psi_1(t)dt\}^2$$

$$\left. + 2\lambda(1-\lambda)(1-2\lambda)^2N\theta^2\int J^3(t)dt\int J(t)\Psi_2(t)dt\right] - \frac{(1-2\lambda)^2}{8\lambda(1-\lambda)N} ,$$

$$\bar{\alpha}_4 = -\frac{(1-2\lambda)^2\theta}{12\{\lambda(1-\lambda)N\}^{1/2}}\frac{\int J^3(t)dt\int J^2(t)\Psi_1(t)dt}{\{\int J^2(t)dt\}^{5/2}} ,$$

$$\bar{\alpha}_5 = \frac{(1-2\lambda)^2}{72\lambda(1-\lambda)N}\frac{\{\int J^3(t)dt\}^2}{\{\int J^2(t)dt\}^3} ,$$

(5.9)
$$\bar{K}_1(x) = \Phi(x) - \phi(x)\left[\sum_{k=0}^{5}\bar{\alpha}_kH_k(x)\right.$$

$$+ \tfrac{1}{2}\left(\frac{\lambda(1-\lambda)}{N\int J^2(t)dt}\right)^{\frac{1}{2}}\theta\left\{2\sum_{j=1}^{N}\operatorname{cov}(J(U_{j:N}),\Psi_1(U_{j:N}))\right.$$

$$\left.\left. - \frac{\int J(t)\Psi_1(t)dt}{\int J^2(t)dt}\sum_{j=1}^{N}\sigma^2(J(U_{j:N}))\right\}\right] ,$$

(5.10)
$$\bar{K}_2(x) = \Phi(x) - \phi(x)\left[\sum_{k=0}^{5}\bar{\alpha}_kH_k(x)\right.$$

$$+ \tfrac{1}{2}\left(\frac{\lambda(1-\lambda)}{N\int J^2(t)dt}\right)^{\frac{1}{2}}\theta\left\{2\int_{N^{-1}}^{1-N^{-1}}J'(t)\Psi_1'(t)t(1-t)dt\right.$$

$$\left.\left. - \frac{\int J(t)\Psi_1(t)dt}{\int J^2(t)dt}\int_{N^{-1}}^{1-N^{-1}}(J'(t))^2t(1-t)dt\right\}\right] ,$$

(5.11)
$$\bar{\eta} = -\left(\frac{\lambda(1-\lambda)N}{\int J^2(t)dt}\right)^{\frac{1}{2}}\theta\int J(t)\Psi_1(t)dt ,$$

where all integrals are over (0,1) unless otherwise indicated. We shall show that $\bar{K}_1(x-\bar{\eta})$ and $\bar{K}_2(x-\bar{\eta})$ are expansions for the d.f. of $\{\lambda(1-\lambda)\sum a_j^2\}^{-\frac{1}{2}}T$ for exact scores. Furthermore let

$$(5.12)\qquad \bar{\beta}_0 = \bar{\alpha}_0 - \frac{(1-2\lambda)}{6\{\lambda(1-\lambda)N\}^{1/2}}\frac{\int J^3(t)dt}{\{\int J^2(t)dt\}^{3/2}}(u_\alpha^2-1) + 2\bar{\alpha}_5(2u_\alpha^3-5u_\alpha) - \frac{u_\alpha}{2N}$$
$$- \left\{\frac{(1-6\lambda+6\lambda^2)}{24\lambda(1-\lambda)N}\frac{\int J^4(t)dt}{\{\int J^2(t)dt\}^2} - \frac{(1-2\lambda)^2}{8\lambda(1-\lambda)N}\right\}(u_\alpha^3-3u_\alpha) ,$$

$$\bar{\beta}_1 = \bar{\alpha}_1 + \bar{\alpha}_5(u_\alpha^2-1)^2 - \frac{(1-2\lambda)^2\theta^2}{12}\frac{\int J^3(t)dt\int J(t)\Psi_2(t)dt}{\{\int J^2(t)dt\}^2}(u_\alpha^2-1) ,$$

$$\bar{\beta}_2 = \bar{\alpha}_2 - \bar{\alpha}_4(u_\alpha^2-1) ,$$

$$\bar{\beta}_3 = \bar{\alpha}_3 - 2\bar{\alpha}_5(u_\alpha^2-1) ,$$

$$\bar{\beta}_k = \bar{\alpha}_k \qquad \text{for } k = 4,5 ,$$

$$(5.13)\qquad \bar{\pi}_i(F,\theta) = 1 - \bar{K}_i(u_\alpha-\bar{\eta}) + \phi(u_\alpha-\bar{\eta})\sum_{k=0}^{5}(\bar{\beta}_k-\bar{\alpha}_k)H_k(u_\alpha-\bar{\eta}), \qquad i = 1,2 ,$$

i.e. $\bar{\pi}_i(F,\theta)$ equals $1 - \bar{K}_i(u_\alpha-\bar{\eta})$ with $\bar{\alpha}_k$ replaced by $\bar{\beta}_k$, $k = 0,\ldots,3$.

THEOREM 5.1. *Let* $F \in \mathcal{F}$, $J \in \mathcal{J}$, $a_j = EJ(U_{j:N})$ *for* $j = 1,\ldots,N$, $G(x) = F(x-\theta)$, $0 \le \theta \le DN^{-\frac{1}{2}}$, $\varepsilon \le \lambda \le 1-\varepsilon$ *and* $\varepsilon' \le \alpha \le 1-\varepsilon'$ *for positive* D, ε *and* ε'. *Then, for every fixed* F, J, D, ε *and* ε', *there exist positive numbers* B, δ_1, δ_2,... *such that* $\lim_{N\to\infty}\delta_N = 0$ *and for every* N

$$(5.14)\qquad \sup_x\left|P\left(\frac{T}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}} \le x\right) - \bar{K}_1(x-\bar{\eta})\right| \le \delta_N N^{-1} ,$$

$$(5.15)\qquad \sup_x\left|P\left(\frac{T}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}} \le x\right) - \bar{K}_2(x-\bar{\eta})\right| \le \delta_N N^{-1}$$
$$+ BN^{-3/2}\int_{N^{-1}}^{1-N^{-1}}|J'(t)|(|J'(t)| + |\Psi_1'(t)|)\{t(1-t)\}^{\frac{1}{2}}dt,$$

$$(5.16)\qquad |\pi(F,\theta) - \bar{\pi}_1(F,\theta)| \le \delta_N N^{-1} ,$$

$$(5.17)\qquad |\pi(F,\theta) - \bar{\pi}_2(F,\theta)| \le \delta_N N^{-1} + BN^{-3/2} \int_{N^{-1}}^{1-N^{-1}} |J'(t)|(|J'(t)| + |\Psi_1'(t)|)\{t(1-t)\}^{\frac{1}{2}}dt.$$

PROOF. In the first part of the proof we shall not need requirement (5.4) but only the weaker assumption $\int J^4(t)dt < \infty$. We proceed as in the proof of theorem 4.1 in ABZ (1976), drawing heavily on the results in appendix 2 of ABZ (1976). Note that these results remain valid in the present context even though the definition of the functions Ψ_i is slightly different here. Throughout the proof we shall make use of O and o symbols that are uniform for fixed F, J, D, ε and ε'.

Because $\sum a_j = N\int J(t)dt = 0$ and in view of the remark made at the end of section 3, the assumptions of theorem 4.1 are satisfied. The proof of corollary A2.1 of ABZ (1976) shows that (5.6) and (5.7) imply that

$$(5.18)\qquad \Psi_1'(t) = o(\{t(1-t)\}^{-7/6}) \quad \text{for } t \to 0,1 .$$

Hence, because of (5.7), $\Psi_1''(t) = o(\{t(1-t)\}^{-13/6})$ and $\Psi_1(t) = o(\{t(1-t)\}^{-1/6})$ for $t \to 0,1$. Since $f(F^{-1})$ has a summable derivative Ψ_1 on (0,1), $f(F^{-1})$ must have limits at 0 and 1; as f is positive on R^1, these limits must be equal to 0. It follows that $f(F^{-1}(t)) = o(\{t(1-t)\}^{5/6})$ for $t \to 0,1$. Combining these facts with the identity $\Psi_2'(t) = \Psi_1''(t)f(F^{-1}(t)) + 3\Psi_1(t)\Psi_1'(t)$, we find that

$$(5.19)\qquad \Psi_2'(t) = o(\{t(1-t)\}^{-4/3}) \quad \text{for } t \to 0,1 .$$

Thus the assumptions of theorem 4.2 are also satisfied and we can take the expansions of section 4 as a starting point for proving theorem 5.1.

In $\tilde{\alpha}_0,\ldots,\tilde{\alpha}_5$, $\tilde{\beta}_0,\ldots,\tilde{\beta}_5$ defined by (4.6) and (4.10) we may replace E_F, σ_F^2 and $\psi_i(Z_j)$ by E, σ^2 and $\psi_i(F^{-1}(U_{j:N})) = \Psi_i(U_{j:N})$ without changing anything. Next, arguing as in corollary A2.2 of ABZ (1976), we see that for all sums of the form $\sum a_j^k$ and $\sum a_j^k Eh(U_{j:N})$ occurring in $\tilde{\alpha}_0,\ldots,\tilde{\alpha}_5$, $\tilde{\beta}_0,\ldots,\tilde{\beta}_5$ we may write

$$(5.20)\qquad \frac{1}{N}\sum a_j^k = \int J^k(t)dt + o(1)\ ,$$

$$(5.21)\qquad \frac{1}{N}\sum a_j^k\, Eh(U_{j:N}) = \int J^k(t)h(t)dt + o(1)\ ,$$

and also

$$(5.22)\qquad \frac{1}{N}\sigma^2\Big(\sum a_j\Psi_1(U_{j:N})\Big) = \iint J(s)J(t)\Psi_1'(s)\Psi_1'(t)[s\wedge t - st]dsdt + o(1).$$

We note that $\bar{\alpha}_0,\ldots,\bar{\alpha}_5$, $\bar{\beta}_0,\ldots,\bar{\beta}_5$ are obtained from $\tilde{\alpha}_0,\ldots,\tilde{\alpha}_5$, $\tilde{\beta}_0,\ldots,\tilde{\beta}_5$ by replacing every expression of the form (5.20) - (5.22) by the corresponding integral on the right in (5.20) - (5.22). Since $\int J^2(t)dt > 0$, we know that for those terms in $\tilde{\alpha}_0,\ldots,\tilde{\alpha}_5$, $\tilde{\beta}_0,\ldots,\tilde{\beta}_5$ that are $O(N^{-1})$, this substitution can only introduce errors that are $o(N^{-1})$.

The first terms in $\tilde{\alpha}_0$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ as well as the second term in $\tilde{\beta}_0$ are generally not $O(N^{-1})$ but only $O(N^{-\frac{1}{2}})$, and here the substitution of integrals for sums gives rise to more complicated remainder terms. This creates problems we did not encounter in the one-sample case where certain symmetries prohibit the occurrence of $O(N^{-\frac{1}{2}})$ terms. We have

$$\frac{1}{N}\sum a_j^2 = \int J^2(t)dt - \frac{1}{N}\sum \sigma^2(J(U_{j:N}))\ ,$$

$$\frac{1}{N}\sum a_j^3 = \int J^3(t)dt - \frac{1}{N}\sum \mathrm{cov}(J(U_{j:N}),J^2(U_{j:N})) - \frac{1}{N}\sum EJ(U_{j:N})\sigma^2(J(U_{j:N}))\ ,$$

$$\frac{1}{N}\sum a_j E\Psi_2(U_{j:N}) = \int J(t)\Psi_2(t)dt - \frac{1}{N}\sum \mathrm{cov}(J(U_{j:N}),\Psi_2(U_{j:N}))$$

$$\frac{1}{N}\sum a_j^2 E\Psi_1(U_{j:N}) = \int J^2(t)\Psi_1(t)dt - \frac{1}{N}\sum \mathrm{cov}(J^2(U_{j:N}),\Psi_1(U_{j:N}))$$

$$- \frac{1}{N}\sum E\Psi_1(U_{j:N})\sigma^2(J(U_{j:N}))\ .$$

By (A2.22) in ABZ (1976), $N^{-3/2}\sum \sigma^2(J(U_{j:N})) = o(N^{-1})$. It follows that for $k = 0,\ldots,5$,

$$(5.23)\qquad \tilde{\alpha}_k - \bar{\alpha}_k = o(N^{-1}) + O(M_1),\qquad \tilde{\beta}_k - \bar{\beta}_k = o(N^{-1}) + O(M_1)\ ,$$

$$(5.24)\qquad M_1 = (1-2\lambda)N^{-3/2}\Big[\,\Big|\sum \operatorname{cov}(J(U_{j:N}),\, J^2(U_{j:N}))\Big|$$

$$+\ \Big|\sum EJ(U_{j:N})\sigma^2(J(U_{j:N}))\Big| + \Big|\sum \operatorname{cov}(J(U_{j:N}),\Psi_2(U_{j:N}))\Big|$$

$$+\ \Big|\sum \operatorname{cov}(J^2(U_{j:N}),\, \Psi_1(U_{j:N}))\Big| + \Big|\sum E\Psi_1(U_{j:N})\sigma^2(J(U_{j:N}))\Big|\Big]\,.$$

By (A2.17), (A2.22) and (A2.23) in ABZ (1976) we have

$$(5.25)\qquad \eta = \bar\eta + \tfrac{1}{2}\left(\frac{\lambda(1-\lambda)}{N\int J^2(t)dt}\right)^{\frac12}\theta\Big[2\sum \operatorname{cov}(J(U_{j:N}),\, \Psi_1(U_{j:N}))$$

$$-\ \frac{\int J(t)\Psi_1(t)dt}{\int J^2(t)dt}\sum \sigma^2(J(U_{j:N}))\Big] + o(N^{-1}) = \bar\eta + o(N^{-\frac12}).$$

Hence, uniformly in x,

$$\tilde K(x-\eta) = \Phi(x-\bar\eta) - \phi(x-\bar\eta)\Big[(\eta-\bar\eta) + \sum_{k=0}^{5}\bar\alpha_k H_k(x-\bar\eta)\Big] + o(N^{-1}) + O(M_1)$$

$$= \bar K_1(x-\bar\eta) + o(N^{-1}) + O(M_1),$$

and similarly

$$\tilde\pi(F,\theta) = \bar\pi_1(F,\theta) + o(N^{-1}) + O(M_1).$$

It follows that, in order to prove (5.14) and (5.16), it suffices to show that $M_1 = o(N^{-1})$. Since (5.15) and (5.17) are immediate consequences of (5.14) and (5.16) on the one hand and (A2.22) and (A2.23) in ABZ (1976) on the other, the proof of the theorem will then be complete.

At this point we finally need condition (5.4) rather than the weaker assumption $\int J^4(t)dt < \infty$. Using (5.4), (5.18) and (5.19) and proceeding as in the proof of corollary A2.1 in ABZ (1976), we find that each term of M_1 is

$$(5.26)\qquad o\Big(N^{-3/2}\int_{N^{-1}}^{1-N^{-1}}\{t(1-t)\}^{-3/2}dt\Big) = o(N^{-1})\,. \qquad \square$$

REMARK. In the above we have stressed the fact that the only reason for requiring (5.4) rather than assuming $\int J^4(t)dt < \infty$, is that we have to show that $M_1 = o(N^{-1})$. However, there are special cases of interest where $\int J^4(t)dt < \infty$ suffices. If either $\lambda = \frac{1}{2}$, or f is a symmetric density and J(t) is antisymmetric about $t = \frac{1}{2}$, then $M_1 = 0$. Less trivially, since $\int J^4(t)dt < \infty$ and (5.5) imply that $J'(t) = o(\{t(1-t)\}^{-5/4})$, we can follow the reasoning leading to (5.26) while retaining the factor $(1-2\lambda)$, to arrive at

$$(5.27) \qquad M_1 = o\Big(|1-2\lambda| N^{-3/2} \int_{N^{-1}}^{1-N^{-1}} \{t(1-t)\}^{-7/4} dt\Big) = o(|1-2\lambda| N^{-3/4}) \ .$$

Hence in the special cases where either $\lambda = \frac{1}{2} + O(N^{-\frac{1}{4}})$, or f is a symmetric density and J is antisymmetric about the point $\frac{1}{2}$, the conclusions of theorem 5.1 will hold if condition (5.4) is replaced by the assumption $\int J^4(t)dt < \infty$. Comparison with ABZ (1976) shows that in these special cases the conditions under which theorem 5.1 holds are essentially the same as the conditions of the comparable theorem 4.1 in ABZ (1976) for the one-sample problem. This is not surprising as one may think of the one-sample case under contiguous alternatives as a two-sample situation with $\lambda = \frac{1}{2} + O_p(N^{-\frac{1}{2}})$.

We now turn to the special case $J = -\Psi_1$. For $F \in \mathcal{F}$ we obtain by partial integration

$$(5.28) \qquad \int \Psi_1(t)\Psi_2(t)dt = \tfrac{1}{2} \int \Psi_1^3(t)dt \ ,$$

$$\int \Psi_1^2(t)\Psi_2(t)dt = \frac{2}{3} \int \Psi_1^4(t)dt \ ,$$

$$\int \Psi_1(t)\Psi_3(t)dt = \frac{2}{3} \int \Psi_1^4(t)dt - \int \Psi_2^2(t)dt \ ,$$

$$\iint \Psi_1(s)\Psi_1(t)\Psi_1'(s)\Psi_1'(t)[s\wedge t - st]dsdt = \frac{1}{4} \int \Psi_1^4(t)dt - \frac{1}{4}\Big(\int \Psi_1^2(t)dt\Big)^2 \ .$$

Substitution of $J = -\Psi_1$ and application of (5.28) considerably simplifies the expressions (5.8) and (5.12) for $\bar{\alpha}_k$ and $\bar{\beta}_k$. Note that $\bar{\eta}$ defined by (5.11) reduces to

$$(5.29)\qquad \eta^* = \theta\{\lambda(1-\lambda)N \int \Psi_1^2(t)dt\}^{\frac{1}{2}} .$$

The expressions for $\bar{\alpha}_k$ and $\bar{\beta}_k$ simplify somewhat further if we express θ in terms of η^* throughout. Finally we rearrange the terms in $\sum \bar{\alpha}_k H_k(x-\eta^*)$ and $\sum \bar{\beta}_k H_k(u_\alpha-\eta^*)$ according to the integrals involved and substitute the explicit expressions (2.43) for the Hermite polynomials H_k. In this way we find after laborious but straightforward calculations that for $J = -\Psi_1'$,

$$(5.30)\qquad \Phi(x-\bar{\eta}) - \phi(x-\bar{\eta}) \sum_{k=0}^{5} \bar{\alpha}_k H_k(x-\bar{\eta}) = L_0(x) ,$$

$$1 - \Phi(u_\alpha-\bar{\eta}) + \phi(u_\alpha-\bar{\eta}) \sum_{k=0}^{5} \bar{\beta}_k H_k(u_\alpha-\bar{\eta}) = \pi_0^*(F,\theta) ,$$

where

$$(5.31)\qquad L_0(x) = \Phi(x-\eta^*)$$

$$- \frac{\phi(x-\eta^*)}{288} \Bigg[\frac{24(1-2\lambda)}{\{\lambda(1-\lambda)N\}^{1/2}} \frac{\int\Psi_1^3(t)dt}{\{\int\Psi_1^2(t)dt\}^{3/2}} \{-2(x^2-1) - 2\eta^* x + \eta^{*2}\}$$

$$+ \frac{4}{\lambda(1-\lambda)N} \frac{\int\Psi_1^4(t)dt}{\{\int\Psi_1^2(t)dt\}^2} \{3(1-6\lambda+6\lambda^2)(x^3-3x+\eta^*(x^2-1)) - 3(1-5\lambda+5\lambda^2)\eta^{*2}x$$

$$+ 5(1-3\lambda+3\lambda^2)\eta^{*3}\} - \frac{48}{\lambda(1-\lambda)N} \frac{\int\Psi_2^2(t)dt}{\{\int\Psi_1^2(t)dt\}^2} (1-3\lambda+3\lambda^2)\eta^{*3}$$

$$+ \frac{(1-2\lambda)^2}{\lambda(1-\lambda)N} \frac{\{\int\Psi_1^3(t)dt\}^2}{\{\int\Psi_1^2(t)dt\}^3} \{4(x^5-10x^3+15x) + 4\eta^*(x^4-6x^2+3) - 8\eta^{*2}(x^3-3x)$$

$$- 4\eta^{*3}(x^2-1) + 5\eta^{*4}x - \eta^{*5}\} + \frac{144x}{N} + \frac{36}{\lambda(1-\lambda)N} \{-(1-2\lambda)^2(x^3-3x+\eta^* x^2)$$

$$+ \eta^* + (1-5\lambda+5\lambda^2)\eta^{*2}x + (1-3\lambda+3\lambda^2)\eta^{*3}\} \Bigg] ,$$

$$\pi_0^*(F,\theta) = 1 - \Phi(u_\alpha-\eta^*) +$$

$$+\frac{\eta^*\phi(u_\alpha-\eta^*)}{2\delta}\left[\frac{2\delta(1-2\lambda)}{\{\lambda(1-\lambda)N\}^{1/2}}\frac{\int\psi_1^3(t)dt}{\{\int\psi_1^2(t)dt\}^{3/2}}(-2u_\alpha+\eta^*)\right.$$

$$+\frac{\delta}{\lambda(1-\lambda)N}\frac{\int\psi_1^4(t)dt}{\{\int\psi_1^2(t)dt\}^2}\{3(1-6\lambda+6\lambda^2)(u_\alpha^2-1)-3(1-5\lambda+5\lambda^2)\eta^*u_\alpha$$

$$-5(1-3\lambda+3\lambda^2)\eta^{*2}\}-\frac{48}{\lambda(1-\lambda)N}\frac{\int\psi_2^2(t)dt}{\{\int\psi_1^2(t)dt\}^2}(1-3\lambda+3\lambda^2)\eta^{*2}$$

$$-\frac{(1-2\lambda)^2}{\lambda(1-\lambda)N}\frac{\{\int\psi_1^3(t)dt\}^2}{\{\int\psi_1^2(t)dt\}^3}\{-8(2u_\alpha^2-1)+4\eta^*(u_\alpha^3+3u_\alpha)-8\eta^{*2}(u_\alpha^2-1)$$

$$\left.+5\eta^{*3}u_\alpha-\eta^{*4}\}+\frac{3\delta}{\lambda(1-\lambda)N}\{-(1-2\lambda)^2u_\alpha^2+1+(1-5\lambda+5\lambda^2)\eta^*u_\alpha\right.$$

$$\left.+(1-3\lambda+3\lambda^2)\eta^{*2}\}\right].$$

Define

$$(5.32)\qquad L_1(x)=L_0(x)+\frac{\eta^*\phi(x-\eta^*)}{2N\int\psi_1^2(t)dt}\sum_{j=1}^{N}\sigma^2(\psi_1(U_{j:N}))\ ,$$

$$L_2(x)=L_0(x)+\frac{\eta^*\phi(x-\eta^*)}{2N\int\psi_1^2(t)dt}\int_{N^{-1}}^{1-N^{-1}}(\psi_1'(t))^2t(1-t)dt\ ,$$

$$L_3(x)=L_0(x)+\frac{\eta^*\phi(x-\eta^*)}{2N\int\psi_1^2(t)dt}\sum_{j=1}^{N}E(\psi_1(U_{j:N})-\psi_1(\tfrac{j}{N+1}))^2\ ,$$

$$\pi_1^*(F,\theta)=\pi_0^*(F,\theta)-\frac{\eta^*\phi(u_\alpha-\eta^*)}{2N\int\psi_1^2(t)dt}\sum_{j=1}^{N}\sigma^2(\psi_1(U_{j:N}))\ ,$$

$$\pi_2^*(F,\theta)=\pi_0^*(F,\theta)-\frac{\eta^*\phi(u_\alpha-\eta^*)}{2N\int\psi_1^2(t)dt}\int_{N^{-1}}^{1-N^{-1}}(\psi_1'(t))^2t(1-t)dt\ ,$$

$$\pi_3^*(F,\theta)=\pi_0^*(F,\theta)-\frac{\eta^*\phi(u_\alpha-\eta^*)}{2N\int\psi_1^2(t)dt}\sum_{j=1}^{N}E(\psi_1(U_{j:N})-\psi_1(\tfrac{j}{N+1}))^2$$

Note that (5.9), (5.10), (5.11), (5.12), (5.30) and (5.31) imply that for $J=-\psi_1$, $\bar{K}_i(x-\bar{\eta})=L_i(x)$ and $\bar{\pi}_i(F,\theta)=\pi_i^*(F,\theta)$ for $i=1,2$. The expansions

L_3 and π_3^* are connected only with approximate scores that were not considered so far.

THEOREM 5.2. *Let* $F \in \mathcal{F}$, $J = -\Psi_1'$, $G(x) = F(x-\theta)$, $0 \le \theta \le DN^{-\frac{1}{2}}$, $\varepsilon \le \lambda \le 1-\varepsilon$ *and* $\varepsilon' \le \alpha \le 1-\varepsilon'$ *for positive* D, ε *and* ε'. *Then, for every fixed* F, D, ε *and* ε', *there exist positive numbers* B, δ_1, $\delta_2, \ldots$ *with* $\lim_{N\to\infty} \delta_N = 0$ *such that the following statements hold for every* N.

(i) *For exact scores* $a_j = -E\Psi_1(U_{j:N})$,

$$\sup_x \left| P\left(\frac{T}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}} \le x \right) - L_1(x) \right| \le \delta_N N^{-1}, \tag{5.33}$$

$$\sup_x \left| P\left(\frac{T}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}} \le x \right) - L_2(x) \right| \le \delta_N N^{-1} + BN^{-3/2} \int_{N^{-1}}^{1-N^{-1}} (\Psi_1'(t))^2 \{t(1-t)\}^{\frac{1}{2}} dt, \tag{5.34}$$

$$|\pi(F,\theta) - \pi_1^*(F,\theta)| \le \delta_N N^{-1}, \tag{5.35}$$

$$|\pi(F,\theta) - \pi_2^*(F,\theta)| \le \delta_N N^{-1} + BN^{-3/2} \int_{N^{-1}}^{1-N^{-1}} (\Psi_1'(t))^2 \{t(1-t)\}^{\frac{1}{2}} dt; \tag{5.36}$$

(ii) *For approximate scores* $a_j = -\Psi_1'\left(\frac{j}{N+1}\right)$,

$$\sup_x \left| P\left(\frac{T-\lambda\Sigma a_j}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}} \le x \right) - L_3(x) \right| \le \delta_N N^{-1}, \tag{5.37}$$

$$\sup_x \left| P\left(\frac{T-\lambda\Sigma a_j}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}} \le x \right) - L_2(x) \right| \le \delta_N N^{-1} + BN^{-3/2} \int_{N^{-1}}^{1-N^{-1}} (\Psi_1'(t))^2 \{t(1-t)\}^{\frac{1}{2}} dt, \tag{5.38}$$

$$|\pi(F,\theta) - \pi_3^*(F,\theta)| \le \delta_N N^{-1} \tag{5.39}$$

and (5.36) *continues to hold.*

PROOF. For $F \in \mathcal{F}$, Ψ_1 is not constant on $(0,1)$, $\int \Psi_1(t)dt = 0$ and Ψ_1^6 is summable. In view of the remark following definition 5.1, this implies that $J \in \mathcal{J}$. We have already noted that $K_i(x-\bar{\eta}) = L_i(x)$ and $\bar{\pi}_i(F,\theta) = \pi_i^*(F,\theta)$ for $i = 1,2$, if $J = -\Psi_1$. Part (i) of the theorem is therefore an immediate consequence of theorem 5.1.

To prove part (ii) we retrace the proof of theorem 5.1 for $J = -\Psi_1$ and approximate scores $a_j = -\Psi_1(j/(N+1))$. The first difficulty we encounter is that in general $\sum a_j \neq 0$. However, lemma A2.3 of ABZ (1976), (5.7) and (5.18) yield

$$(5.40) \qquad a_\cdot = \frac{1}{N}\sum_{j=1}^{N} a_j = -\int_0^1 \Psi_1(t)dt + O(N^{-1}\int_{N^{-1}}^{1-N^{-1}} |\Psi_1'(t)|dt) = o(N^{-5/6}) ,$$

and one easily verifies that the conditions of theorem 4.1 hold for the reduced scores $a_j - a_\cdot$. Since the assumptions of theorem 4.2 are also satisfied, we have

$$(5.41) \qquad \sup_x \left| P\left(\frac{T-\lambda\Sigma a_j}{\{\lambda(1-\lambda)\Sigma(a_j-a_\cdot)^2\}^{\frac{1}{2}}} \le x \right) - \hat{K}(x-\hat{\eta}) \right| = O(N^{-5/4}) ,$$

where $\hat{K}$ and $\hat{\eta}$ are obtained from $\tilde{K}$ and η by replacing a_j by $a_j - a_\cdot$ throughout. Because, by (3.10) and (5.40),

$$(5.42) \qquad \sum (a_j-a_\cdot)^2 = \sum a_j^2(1+o(N^{-5/3})) ,$$

we can change the norming constant $\sum(a_j-a_\cdot)^2$ of T in (5.41) back to $\sum a_j^2$ with impunity. As $\int \Psi_1(t)dt = 0$, (5.42) also ensures that $|\hat{\eta} - \eta| = o(N^{-5/3})$. Finally (A2.16) of ABZ (1976) and (5.18) imply that $\sigma_F^2(\sum a_j\psi_1(Z_j)) = O(N)$ for $J = -\Psi_1$ and, together with (5.42), (3.10), (5.6) and (5.40), this yields $\sup_x|\hat{K}(x) - \tilde{K}(x)| = o(N^{-4/3})$. Combining these results we find

$$(5.43) \qquad \sup_x \left| P\left(\frac{T-\lambda\Sigma a_j}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}} \le x \right) - \tilde{K}(x-\eta) \right| = O(N^{-5/4})$$

and similarly

$$(5.44) \qquad |\pi(F,\theta) - \tilde{\pi}(F,\theta)| = O(N^{-5/4}) .$$

The remainder of the proof parallels that of theorem 5.1 for the special case $J = -\psi_1$. We replace all sums as well as $\sigma^2(\sum a_j \psi_1(U_{j:N}))$ by the appropriate integrals. The reasoning of corollary A2.2 of ABZ (1976) shows that for those terms in the expansions that are $O(N^{-1})$, this substitution will only lead to errors that are $o(N^{-1})$. For the $O(N^{-\frac{1}{2}})$ terms the error committed is $O(M_1) + O(M_2)$, where M_1 is given by (5.24) with $J = -\psi_1$ and M_2 originates from the difference between exact and approximate scores. It was shown in the proof of theorem 5.1 that $M_1 = o(N^{-1})$. With regard to M_2, (5.7), lemma A2.3 of ABZ (1976), (5.18) and (5.19) imply that, uniformly in j,

$$(5.45)\qquad \left|\{E\psi_1(U_{j:N})\}^k - \psi_1^k\left(\frac{j}{N+1}\right)\right| = O(N^{-1}) + o\left(N^{-1}\left\{\frac{j(N-j+1)}{(N+1)^2}\right\}^{-1-k/6}\right)$$

$$\left|E\psi_1(U_{j:N})\right| = O(1) + o\left(\left\{\frac{j(N-j+1)}{(N+1)^2}\right\}^{-1/6}\right),$$

$$\left|E\psi_2(U_{j:N})\right| = O(1) + o\left(\left\{\frac{j(N-j+1)}{(N+1)^2}\right\}^{-1/3}\right),$$

where $k = 1,2,3$. It follows that M_2 is of the form (5.26) and is therefore $o(N^{-1})$.

It remains to replace η by η^*. Because of (5.7), (5.18) and lemma A2.3 of ABZ (1976), $N^{-1}\sum \sigma^2(\psi_1(U_{j:N})) = o(N^{-2/3})$, and in view of (5.45),

$$\frac{1}{N}\sum \psi_1\left(\frac{j}{N+1}\right)E\psi_1(U_{j:N}) - \int \psi_1^2(t)dt$$

$$= -\frac{1}{N}\sum E\psi_1(U_{j:N})\left[\psi_1(U_{j:N}) - \psi_1\left(\frac{j}{N+1}\right)\right] = o(N^{-2/3}),$$

$$\frac{1}{N}\sum \psi_1^2\left(\frac{j}{N+1}\right) - \int \psi_1^2(t)dt = -\frac{1}{N}\sum\left[E\psi_1^2(U_{j:N}) - \psi_1^2\left(\frac{j}{N+1}\right)\right] = o(N^{-2/3}).$$

Hence, for $J = -\psi_1$,

$$(5.46)\qquad \eta = \eta^* - \frac{\eta^*}{2N\int\psi_1^2(t)dt}\sum E\{\psi_1(U_{j:N}) - \psi_1\left(\frac{j}{N+1}\right)\}^2 + o(N^{-4/3})$$

$$= \eta^* + o(N^{-2/3}),$$

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and a comparison with (5.25) for $J = -\Psi_1$ shows that (5.37) and (5.39) will hold if L_3 and π_3^* can be obtained from L_1 and π_1^* by replacing $\sum \sigma^2(\Psi_1(U_{j:N}))$ by $\sum E\{\Psi_1(U_{j:N}) - \Psi_1(j/(N+1))\}^2$. Since this is true, (5.37) and (5.39) are proved. The validity of (5.38) and (5.36) for approximate scores is a consequence of (5.37), (5.39) and corollary A2.2 of ABZ (1976). The proof of the theorem is complete. □

At this point it is appropriate to repeat some remarks made in ABZ (1976). The correspondence between expansions (5.34) and (5.38) and the fact that (5.36) holds for both exact and approximate scores seem to be typical for the case $J = -\Psi_1$. In the general case where $J \neq -\Psi_1$, expansions (5.15) and (5.17) will not hold for approximate scores even if T is replaced by $T - \lambda \sum a_j$ in (5.15). A second remark is that the growth conditions on J' and Ψ_1' implicit in our assumptions (viz. (5.4) and (5.18)) do not guarantee that the right-hand side in (5.15), (5.17), (5.34), (5.36) and (5.38) is indeed $o(N^{-1})$ as is our aim. For this we would need $J'(t) = o(\{t(1-t)\}^{-1})$ and $\Psi_1'(t) = o(\{t(1-t)\}^{-1})$. This may explain the presence of the remaining expansions in theorems 5.1 and 5.2, which are less explicit but do have remainder $o(N^{-1})$ under the conditions stated. Note that their presence in theorem 5.2 also indicates that even for $J = -\Psi_1$, expansions for exact and approximate scores are not necessarily identical to $o(N^{-1})$. Finally we should point out that similar expansions with remainder $o(N^{-1})$ might have been given in theorem 4.2 of ABZ (1976) where they were unfortunately omitted.

We conclude this section with a few examples of the power expansions in theorems 5.1 and 5.2. First we consider the powers $\pi_W(\Phi,\theta)$ and $\pi_W(\Lambda,\theta)$ of Wilcoxon's two-sample test (W) against normal and logistic location alternatives $(\Phi(x),\Phi(x-\theta))$ and $(\Lambda(x),\Lambda(x-\theta))$ respectively, where $\Lambda(x) = (1+\exp\{-x\})^{-1}$ and $\theta = O(N^{-\frac{1}{2}})$. We find

$$(5.47)\qquad \pi_W(\Phi,\theta) = 1 - \Phi(u_\alpha - \bar{\eta}) + \frac{\bar{\eta}\phi(u_\alpha - \bar{\eta})}{N}\left[-\tfrac{1}{2} - \frac{37-217\lambda+217\lambda^2}{20\lambda(1-\lambda)}(u_\alpha^2-1)\right.$$

$$\left. + \frac{1}{\lambda(1-\lambda)}\left\{\frac{\sqrt{3}}{6} + \frac{67-437\lambda+437\lambda^2}{20}\right\}u_\alpha\bar{\eta} + \right.$$

$$- \frac{1}{\lambda(1-\lambda)} \left\{ \frac{\sqrt{3}}{6} + \frac{\pi}{36} + \frac{29-219\lambda+219\lambda^2}{20} \right\} \bar{\eta}^2$$

$$+ \frac{(1-6\lambda+6\lambda^2)}{\lambda(1-\lambda)} \frac{6 \arctn\sqrt{2}}{\pi} \{u_\alpha^2 - 1 - 2u_\alpha \bar{\eta} + \bar{\eta}^2\} \Big] + o(N^{-1}),$$

where $\bar{\eta} = \left(\frac{3\lambda(1-\lambda)N}{\pi}\right)^{\frac{1}{2}} \theta$, and

$$(5.48) \qquad \pi_W(\Lambda,\theta) = 1 - \Phi(u_\alpha - \eta^*) + \frac{\eta^* \phi(u_\alpha - \eta^*)}{N} \Big[- \tfrac{1}{2} - \frac{1-\lambda+\lambda^2}{20\lambda(1-\lambda)} (u_\alpha^2 - 1)$$

$$+ \frac{1-5\lambda+5\lambda^2}{20\lambda(1-\lambda)} u_\alpha \eta^* - \frac{1-3\lambda+3\lambda^2}{20\lambda(1-\lambda)} \eta^{*2} \Big] + o(N^{-1}),$$

where $\eta^* = \left(\frac{\lambda(1-\lambda)N}{3}\right)^{\frac{1}{2}} \theta$.

As a second example we compute expansions for the powers $\pi_{NS}(\Phi,\theta)$ and $\pi_{NS}(\Lambda,\theta)$ of the two-sample normal scores test against the normal and logistic location alternatives described above. One of the integrals occurring in this computation is

$$(5.49) \qquad \int_{N^{-1}}^{1-N^{-1}} \frac{t(1-t)}{\{\phi(\Phi^{-1}(t))\}^2} \, dt = 2 \int_0^{\Phi^{-1}(1-N^{-1})} \frac{\Phi(x)(1-\Phi(x))}{\phi(x)} \, dx$$

and since its asymptotic evaluation is not entirely trivial, we provide some details. Let γ denote Euler's constant

$$(5.50) \qquad \gamma = \lim_{k\to\infty} \left\{ \sum_{j=1}^{k} \frac{1}{j} - \log k \right\} = 0.577216 \ldots$$

and note that (cf. RYSHIK and GRADSTEIN (1957), p.197)

$$(5.51) \qquad \int_0^\infty e^{-u} \log u \, du = -\gamma,$$

$$(5.52) \qquad \int_0^\infty \phi(u) \log u \, du = -\tfrac{1}{4} \log 2 - \tfrac{1}{4} \gamma .$$

To evaluate (5.49) we begin by writing for $z > 0$

$$(5.53)\qquad \int_0^z \frac{1-\Phi(x)}{\phi(x)}\,dx = \int_0^z dx \int_x^\infty e^{-\frac{1}{2}(y-x)(y+x)}\,dy$$

$$= \tfrac{1}{2}\int_0^\infty du \int_u^{2z+u} e^{-\frac{1}{2}uv}\,dv = \int_0^\infty \frac{1}{u}\,e^{-\frac{1}{2}u^2}(1-e^{-zu})\,du.$$

It follows from (5.53) and (5.51) that for $z \to \infty$,

$$(5.54)\qquad \int_0^z \frac{1-\Phi(x)}{\phi(x)}\,dx = \int_0^{z^{-\frac{1}{2}}} \frac{1}{u}(1-e^{-zu})\,du + \int_{z^{-\frac{1}{2}}}^\infty \frac{1}{u}\,e^{-\frac{1}{2}u^2}\,du + o(1)$$

$$= \int_0^{z^{\frac{1}{2}}} \frac{1}{u}(1-e^{-u})\,du + \tfrac{1}{2}\int_{(2z)^{-1}}^\infty \frac{1}{u}\,e^{-u}\,du + o(1)$$

$$= \tfrac{1}{2}\log z - \int_0^{z^{\frac{1}{2}}} e^{-u}\log u\,du + \tfrac{1}{2}\log(2z) + \tfrac{1}{2}\int_{(2z)^{-1}}^\infty e^{-u}\log u\,du + o(1)$$

$$= \log z + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\gamma + o(1).$$

Similarly (5.53), (5.51) and (5.52) imply that

$$(5.55)\qquad \int_0^\infty \frac{(1-\Phi(x))^2}{\phi(x)}\,dx = \int_0^\infty \phi(x)\,dx \int_0^\infty \frac{1}{u}\,e^{-\frac{1}{2}u^2}(1-e^{-xu})\,du$$

$$= \int_0^\infty \frac{1}{u}\left\{\tfrac{1}{2}e^{-\frac{1}{2}u^2} - (1-\Phi(u))\right\}du = \int_0^\infty \log u\left\{\tfrac{1}{2}ue^{-\frac{1}{2}u^2} - \phi(u)\right\}du$$

$$= \tfrac{1}{4}\log 2 + \tfrac{1}{4}\int_0^\infty e^{-x}\log x\,dx - \int_0^\infty \phi(u)\log u\,du = \tfrac{1}{2}\log 2.$$

Since $\log \Phi^{-1}(1-N^{-1}) = \tfrac{1}{2}\log\log N + \tfrac{1}{2}\log 2 + o(1)$ for $N \to \infty$, (5.49), (5.54) and (5.55) imply that

$$(5.56)\qquad \int_{N^{-1}}^{1-N^{-1}} \frac{t(1-t)}{\{\phi(\Phi^{-1}(t))\}^2}\,dt = \log\log N + \log 2 + \gamma + o(1).$$

With the aid of (5.56) we find

$$\pi_{NS}(\Phi,\theta) = 1 - \Phi(u_\alpha - \eta^*) + \frac{\eta^*\phi(u_\alpha-\eta^*)}{N}\left[-\tfrac{1}{2}\log\log N - \tfrac{1}{2}\log 2 - \tfrac{1}{2}\gamma + \tfrac{1}{2} - \tfrac{1}{4}(u_\alpha^2-1)\right] + o(N^{-1}), \tag{5.57}$$

where now $\eta^* = \{\lambda(1-\lambda)N\}^{\frac{1}{2}}\theta$, and

$$\begin{aligned}\pi_{NS}(\Lambda,\theta) = {} & 1 - \Phi(u_\alpha - \bar{\eta}) + \frac{\bar{\eta}\phi(u_\alpha-\bar{\eta})}{N}\left[\tfrac{1}{2}\log\log N + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\gamma\right. \\ & - \frac{3}{2} - \frac{1-3\lambda+3\lambda^2}{12\lambda(1-\lambda)}(u_\alpha^2-1) + \left\{\frac{\sqrt{3}(1-2\lambda)^2}{4\lambda(1-\lambda)} - \frac{\pi}{6} - \frac{4-21\lambda+21\lambda^2}{12\lambda(1-\lambda)}\right\}u_\alpha\bar{\eta} \\ & \left. + \left\{\frac{6(1-5\lambda+5\lambda^2)}{\lambda(1-\lambda)}\operatorname{arctn}\sqrt{2} - \frac{\sqrt{3}(1-2\lambda)^2}{4\lambda(1-\lambda)} - \frac{11\pi(1-5\lambda+5\lambda^2)}{6\lambda(1-\lambda)} + \frac{5-21\lambda+21\lambda^2}{12\lambda(1-\lambda)}\right\}\bar{\eta}^2\right] \\ & + o(N^{-1}),\end{aligned} \tag{5.58}$$

where now $\bar{\eta} = \{\lambda(1-\lambda)N/\pi\}^{\frac{1}{2}}\theta$. Note that theorem 5.2 ensures that expansion (5.57) is also valid for van der Waerden's two-sample test which is based on the approximate scores $a_j = \Phi^{-1}(j/(N+1))$.

6. THE PERMUTATION TEST BASED ON THE SAMPLE MEANS

In ABZ (1976) two results were given for permutation tests in the one-sample problem. The first of these is an asymptotic expansion for the power of the locally most powerful permutation test against contiguous shift alternatives. Secondly it was shown that the difference between the powers of the permutation test based on the sum of transformed observations $\sum j(X_i)$ and Student's test applied to $j(X_1),\ldots,j(X_N)$ is $o(N^{-1})$ for a large class of alternatives.

In the present paper we shall forego the two-sample analogus of the first mentioned result; the expansion can be obtained in a straightforward manner in much the same way as in the one-sample case but the computations will be extremely tedious. We shall concentrate on the comparison with Student's test. For simplicity we take j to be the identity, thus comparing the two-sample permutation test based on the sample means with Student's two-sample test. Also, we restrict attention to contiguous location alternatives.

As before, we assume that $X_1,\ldots,X_N$ are independent, $X_1,\ldots,X_m$ having common d.f. F and $X_{m+1},\ldots,X_N$ having common d.f. $G(x) = F(x-\theta)$; $Z = (Z_1,\ldots,Z_N)$ denotes the vector of order statistics of $X_1,\ldots,X_N$; We wish to test the hypothesis $\theta = 0$ against the alternative $\theta > 0$ at a fixed level $\alpha \in (0,1)$. We denote probabilities and expectations under the alternative by P and E, and under the hypothesis by P_F and E_F. Note that we donot assume that F has a density, as we did in the previous sections.

The permutation test rejects the hypothesis if

$$\sum_{i=m+1}^{N} X_i \geq \xi_\alpha(Z), \tag{6.1}$$

possibly with randomization if equality occurs. Here $\xi_\alpha(Z)$ is chosen in such a way that

$$P_F\left(\sum_{i=m+1}^{N} X_i \geq \xi_\alpha \mid Z \right) = \alpha \quad \text{a.s.} \tag{6.2}$$

with an obvious modification if there is randomization. If F is known, Student's test rejects the hypothesis if

$$(6.3)\qquad \tilde{T} = \frac{\{\lambda(1-\lambda)N(N-2)\}^{\frac{1}{2}}(X_\bullet^{(2)}-X_\bullet^{(1)})}{[\sum_{i=1}^{m}(X_i-X_\bullet^{(1)})^2 + \sum_{i=m+1}^{N}(X_i-X_\bullet^{(2)})^2]^{\frac{1}{2}}} \geq t_\alpha ,$$

where

$$X_\bullet^{(1)} = \frac{1}{m}\sum_{i=1}^{m} X_i, \qquad X_\bullet^{(2)} = \frac{1}{n}\sum_{i=m+1}^{N} X_i .$$

Here t_α depends on F,N,λ and α and is chosen in such a way that the test has level α. Again there may be randomization. Let $\pi_{Pe}(F,\theta)$ and $\pi_{St}(F,\theta)$ denote the power against the alternative $(F,F(\cdot-\theta))$ of the tests (6.1) and (6.3) respectively.

THEOREM 6.1. *Suppose that positive numbers* $c,C,D,\varepsilon,\varepsilon',\delta$ *and* $r > 8$ *exist such that* F^{-1} *is differentiable on an interval of length at least* δ *where*

$$(6.4)\qquad \frac{d}{dt}F^{-1}(t) \geq c,$$

and such that $\int|x|^r dF(x) \leq C$, $0 \leq \theta \leq DN^{-\frac{1}{2}}$, $\varepsilon \leq \lambda \leq 1-\varepsilon$ *and* $\varepsilon' \leq \alpha \leq 1-\varepsilon'$. *Then there exist* $B > 0$ *depending only on* $c,C,D,\varepsilon,\varepsilon'$, *and* δ, *and* $\beta > 0$ *depending only on* r *such that*

$$(6.5)\qquad |\pi_{Pe}(F,\theta) - \pi_{St}(F,\theta)| \leq BN^{-1-\beta}.$$

PROOF. We shall draw heavily on the proof of theorem 5.2 in ABZ (1976). The only essentially new problem is caused again by the occurrence of a term of order $N^{-\frac{1}{2}}$ in the expansions. The O symbols in this proof are uniform for fixed $c,C,D,\varepsilon,\varepsilon'$ and δ. Since both tests are location invariant we may assume without loss of generality that $\int x dF(x) = 0$.

We begin by collecting some results on moments that will be needed throughout the proof. Define

$$(6.6)\qquad \beta = \min\left(\frac{r-8}{2r+8}, \frac{1}{4}\right),$$

$$(6.7)\qquad X_\bullet = Z_\bullet = \frac{1}{N}\sum_{i=1}^{N} X_i = \frac{1}{N}\sum_{j=1}^{N} Z_j$$

and note that $EX_N^k = E(X_1+\theta)^k$ and that $EX_1 = 0$. Proceeding as in the proof of

theorem 5.2 in ABZ (1976) we see that

$$(6.8) \qquad \frac{c^2\delta^3}{12} \le EX_1^2 \le [EX_1^4]^{\frac{1}{2}} \le C^{2/r},$$

and that, uniformly on a set of probability $1 - O(N^{-1-\beta})$ under P as well as under P_F,

$$(6.9) \qquad \frac{1}{N}\sum_{i=1}^{m} X_i^k = (1-\lambda)EX_1^k + O(N^{-\beta}) \quad , \qquad k = 1,\ldots,4,$$

$$(6.10) \qquad \frac{1}{N}\sum_{i=m+1}^{N} X_i^k = \lambda EX_1^k + O(N^{-\beta}) \quad , \qquad k = 1,\ldots,4,$$

$$(6.11) \qquad \frac{1}{N}\sum_{i=1}^{m} (X_i - X_\cdot)^k = (1-\lambda)EX_1^k + O(N^{-\beta}), \qquad k = 2,\ldots,4,$$

$$(6.12) \qquad \frac{1}{N}\sum_{i=m+1}^{N} (X_i - X_\cdot)^k = \lambda EX_1^k + O(N^{-\beta}) \quad , \qquad k = 2,\ldots,4.$$

For $k = 1$, (6.9) and (6.10) are insufficient for our purposes. Arguing as in (5.13) in ABZ (1976) for $\tau = N^{-3/8}$, we find

$$(6.13) \qquad \frac{1}{N}\sum_{i=1}^{m} X_i = O(N^{-3/8}), \qquad \frac{1}{N}\sum_{i=m+1}^{N} X_i = O(N^{-3/8}),$$

uniformly with probability $1 - O(N^{-1-\beta})$ under both P and P_F.

We shall also have to consider the quantity $\ell\{x:\exists_i\ |x-X_i| < \zeta\}$ for some $\zeta \ge N^{-3/2}\log N$, where ℓ denotes Lebesgue measure. Borrowing from the proof of theorem 5.2 in ABZ (1976) again, we find that for $\zeta = N^{-3/2}\log N$,

$$(6.14) \qquad \ell\{x:\exists_i\ |x-X_i| < \zeta\} \ge \frac{\delta N\zeta}{6}$$

with probability $1 - O(N^{-1-\beta})$ both under P and under P_F. Let E_1 be a set on which (6.9)-(6.14) hold uniformly, with $P(E_1) = 1 - O(N^{-1-\beta})$ and $P_F(E_1) = 1 - O(N^{-1-\beta})$.

Under the hypothesis P_F and conditional on Z the d.f. of

$$N^{-\frac{1}{2}}\Big(\sum_{i=m+1}^{N} X_i - \lambda \sum_{j=1}^{N} Z_j\Big)$$

equals $R(x,p)$ defined in (2.9) with $p_j = \lambda$ and $a_j = Z_j$ for $j = 1,\ldots,N$. Hence corollary 2.1 provides an expansion for this conditional d.f. that holds uniformly on any set where the $a_j = Z_j$ satisfy (2.61) and (2.62) for some fixed positive c,C, and δ, and in view of (6.8)-(6.14) such a set is contained in E_1. Since $\varepsilon' \le \alpha \le 1-\varepsilon'$, this yields an expansion for $\xi_\alpha(Z)$. We find (cf. (4.14))

$$(6.15)\qquad \frac{\xi_\alpha(Z)-\lambda\sum Z_j}{[\lambda(1-\lambda)\sum(Z_j-Z_\cdot)^2]^{\frac12}} = u_\alpha + \frac{(1-2\lambda)\sum(Z_j-Z_\cdot)^3}{6\{\lambda(1-\lambda)\}^{\frac12}[\sum(Z_j-Z_\cdot)^2]^{3/2}}(u_\alpha^2-1)$$
$$- \frac{(1-2\lambda)^2[\sum(Z_j-Z_\cdot)^3]^2}{36\lambda(1-\lambda)[\sum(Z_j-Z_\cdot)^2]^3}(2u_\alpha^3-5u_\alpha) + \frac{u_\alpha}{2N}$$
$$+ \left\{\frac{(1-6\lambda+6\lambda^2)\sum(Z_j-Z_\cdot)^4}{24\lambda(1-\lambda)[\sum(Z_j-Z_\cdot)^2]^2} - \frac{(1-2\lambda)^2}{8\lambda(1-\lambda)N}\right\}(u_\alpha^3-3u_\alpha) + O(N^{-5/4}),$$

uniformly on E_1.

Next we start to compute under the alternative P. We have $P(E_1) = 1 - O(N^{-1-\beta})$ and on E_1 we can use (6.8), (6.11) and (6.12) to replace the random terms of order $O(N^{-1})$ on the right in (6.15) by constants. In this way we arrive at

$$(6.16)\qquad \pi_{Pe}(F,\theta) = P(T^* \ge \xi_\alpha^* + U_1 + O(N^{-1-\beta})) + O(N^{-1-\beta}),$$

where the first remainder term depends on Z but may now be taken to be uniformly $O(N^{-1-\beta})$, and where

$$(6.17)\qquad T^* = \frac{\sum_{i=m+1}^{N} X_i - \lambda\sum Z_j}{[\lambda(1-\lambda)\sum(Z_j-Z_\cdot)^2]^{\frac12}},$$

$$(6.18)\qquad \xi_\alpha^* = u_\alpha + \frac{(1-2\lambda)EX_1^3}{6\{\lambda(1-\lambda)N\}^{\frac12}(EX_1^2)^{3/2}}(u_\alpha^2-1)$$
$$- \frac{(1-2\lambda)^2(EX_1^3)^2}{36\lambda(1-\lambda)N(EX_1^2)^3}(2u_\alpha^3-5u_\alpha) + \frac{u_\alpha}{2N}$$
$$+ \left\{\frac{(1-6\lambda+6\lambda^2)EX_1^4}{24\lambda(1-\lambda)N(EX_1^2)^2} - \frac{(1-2\lambda)^2}{8\lambda(1-\lambda)N}\right\}(u_\alpha^3-3u_\alpha),$$

$$(6.19)\qquad U_1 = \frac{(1-2\lambda)(u_\alpha^2-1)}{6\{\lambda(1-\lambda)\}^{\frac{1}{2}}}\left\{\frac{\sum(Z_j-Z_\cdot)^3}{[\sum(Z_j-Z_\cdot)^2]^{3/2}} - \frac{N^{-\frac{1}{2}}EX_1^3}{(EX_1^2)^{3/2}}\right\}.$$

The basic problem is now to show that the r.v. U_1 originating from the $O(N^{-\frac{1}{2}})$ term in (6.15), may be omitted in (6.16). Since U_1 is a r.v. of order N^{-1}, this problem is nontrivial. We shall show that because U_1 depends only on Z and is approximately centered, a cancellation occurs which makes its contribution to (6.16) of negligible order. Several methods of proof are possible. We choose one that does not require any additional assumptions.

In (6.16), P may be replaced by P_F if X_i is replaced by $X_i+\theta$ for $i = m+1,\ldots,N$, which transforms T^* and U_1 into $T^*(\theta)$ and $U_1(\theta)$, say. On the set E_1, (6.8)-(6.13) ensure that we can expand $T^*(\theta)$ and $U_1(\theta)$ about T^* and U_1. Replacing r.v.'s by their expected values if the difference is of negligible order, a simple calculation shows that under P_F we have, uniformly on the set E_1,

$$(6.20)\qquad T^*(\theta) = T^*\left[1 - \frac{3\lambda(1-\lambda)N\theta^2}{2\sum(Z_j-Z_\cdot)^2}\right] - T^{*2}\frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}\theta}{[\sum(Z_j-Z_\cdot)^2]^{\frac{1}{2}}}$$

$$+ \frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}N\theta}{[\sum(Z_j-Z_\cdot)^2]^{\frac{1}{2}}} - \frac{\{\lambda(1-\lambda)\}^{3/2}N^{\frac{1}{2}}\theta^3}{2(EX_1^2)^{3/2}} + O(N^{-1-\beta}),$$

$$(6.21)\qquad U_1(\theta) = U_1 + O(N^{-1-\beta}).$$

Another easy calculation where we use (6.8) - (6.13) to bound the terms in (6.20) and (6.21) and to replace r.v.'s by their expected values whenever possible and where we note that $\xi_\alpha^* = u_\alpha + O(N^{-\frac{1}{2}})$, shows that uniformly on E_1, the inequality $T^*(\theta) \geq \xi_\alpha^* + U_1(\theta) + O(N^{-1-\beta})$ is equivalent to $T^* \geq \xi_\alpha^*(\theta) - U_0 + U_1 + O(N^{-1-\beta})$, where

$$(6.22)\qquad \xi_\alpha^*(\theta) = \xi_\alpha^* - \frac{\{\lambda(1-\lambda)N\}^{\frac{1}{2}}\theta}{(EX_1^2)^{\frac{1}{2}}} + \frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}\theta}{(NEX_1^2)^{\frac{1}{2}}}u_\alpha^2 - \frac{\lambda(1-\lambda)\theta^2}{2EX_1^2}u_\alpha,$$

$$(6.23)\qquad U_0 = \{\lambda(1-\lambda)\}^{\frac{1}{2}}N\theta\left\{\frac{1}{[\sum(Z_j-Z_\cdot)^2]^{\frac{1}{2}}} - \frac{1}{(NEX_1^2)^{\frac{1}{2}}}\right\}.$$

Since $P_F(E_1) = 1 - O(N^{-1-\beta})$, this implies

$$\pi_{Pe}(F,\theta) = P_F(T^*(\theta) \geq \xi_\alpha^* + U_1(\theta) + O(N^{-1-\beta})) + O(N^{-1-\beta}) \tag{6.24}$$

$$= P_F(T^* \geq \xi_\alpha^*(\theta) - U_0 + U_1 + O(N^{-1-\beta})) + O(N^{-1-\beta}),$$

where the first remainder term in the last member depends on $X_1,\ldots,X_N$ but is uniformly $O(N^{-1-\beta})$.

Since U_0 and U_1 depend on $X_1,\ldots,X_N$ only through Z, we can compute $P_F(T^* \leq \xi_\alpha^*(\theta) - U_0 + U_1 + O(N^{-1-\beta}))$ by taking the expectation under P_F of the conditional d.f. of T^* given Z under P_F evaluated at the point $\xi_\alpha^*(\theta) - U_0 + U_1 + O(N^{-1-\beta})$. Corollary 2.1 provides an expansion for the conditional d.f. of T^* given Z under P_F that is valid uniformly on E_1, and $P_F(E_1) = 1 - O(N^{-1-\beta})$. Combining these facts and simplifying as much as possible with the aid of (6.8) - (6.13) - note that (6.8), (6.11) and (6.12) imply that $U_0 = O(N^{-\beta})$ and $U_1 = O(N^{-\frac{1}{2}-\beta})$ - we find

$$\pi_{Pe}(F,\theta) = 1 - E_F\Phi(\xi_\alpha^*(\theta) - U_0 + U_1) \tag{6.25}$$

$$+ \frac{(1-2\lambda)}{6\{\lambda(1-\lambda)\}^{\frac{1}{2}}} E_F\left[\frac{\sum(Z_j - Z_\cdot)^3}{[\sum(Z_j - Z_\cdot)^2]^{3/2}} \phi(\xi_\alpha^*(\theta) - U_0) H_2(\xi_\alpha^*(\theta) - U_0)\right]$$

$$+ \phi(\xi_\alpha^*(\theta))\left[\frac{1}{2N}\xi_\alpha^*(\theta) + \left\{\frac{(1-6\lambda+6\lambda^2)}{24\lambda(1-\lambda)N}\frac{EX_1^4}{(EX_1^2)^2} - \frac{(1-2\lambda)^2}{8\lambda(1-\lambda)N}\right\}H_3(\xi_\alpha^*(\theta))\right.$$

$$\left. + \frac{(1-2\lambda)^2}{72\lambda(1-\lambda)N}\frac{(EX_1^3)^2}{(EX_1^2)^3} H_5(\xi_\alpha^*(\theta))\right] + O(N^{-1-\beta}).$$

Thus we see that the contribution of U_1 to the expansion for $\pi_{Pe}(F,\theta)$ is restricted to its contribution to $-E_F\Phi(\xi_\alpha^*(\theta) - U_0 + U_1)$. On the set E_1 we have $U_1 = \tilde{U}_1 + M$, where

$$\tilde{U}_1 = \frac{(1-2\lambda)(u_\alpha^2 - 1)}{6\{\lambda(1-\lambda)N\}^{\frac{1}{2}}}\left\{\left(\frac{\sum(Z_j - Z_\cdot)^3}{NEX_1^3} - 1\right) - \frac{3}{2}\left(\frac{\sum(Z_j - Z_\cdot)^2}{NEX_1^2} - 1\right)\right\},$$

and

$$N^{\frac{1}{2}}M = O\left(\left\{\frac{\sum(Z_j - Z_\cdot)^2}{NEX_1^2} - 1\right\}^2 + \left|\frac{\sum(Z_j - Z_\cdot)^2}{NEX_1^2} - 1\right| \cdot \left|\frac{\sum(Z_j - Z_\cdot)^3}{NEX_1^3} - 1\right|\right)$$

uniformly on E_1; also $U_0 = O(N^{-\beta})$ uniformly on E_1. Let χ_{E_1} denote the indicator of E_1. Then, because $P_F(E_1) = 1-O(N^{-1-\beta})$,

$$E_F\Phi(\xi_\alpha^*(\theta)-U_0+U_1) = E_F\Phi(\xi_\alpha^*(\theta)-U_0\chi_{E_1}+\tilde{U}_1+M\chi_{E_1}) + O(N^{-1-\beta})$$

$$= E_F\Phi(\xi_\alpha^*(\theta)-U_0\chi_{E_1}) + E_F\phi(\xi_\alpha^*(\theta)-U_0\chi_{E_1})\tilde{U}_1 + O(N^{-1-\beta}+E_F\{\tilde{U}_1^2+|M|\chi_E\})$$

$$= E_F\Phi(\xi_\alpha^*(\theta)-U_0) + \phi(\xi_\alpha^*(\theta))E_F\tilde{U}_1 + O(N^{-1-\beta}+E_F\{N^{-\beta}|\tilde{U}_1|+\tilde{U}_1^2+|M|\chi_E\}).$$

Noting that $\sum(Z_j-Z_\cdot)^k = \sum(X_i-X_\cdot)^k$, $E_FX_i = 0$ and $E_F|X_i|^r \le C$ for some $r > 8$, one easily verifies that $E_F\tilde{U}_1 = O(N^{-3/2})$, $E_F\tilde{U}_1^2 = O(N^{-2})$ and $E_F|M|\chi_E = O(N^{-3/2})$. It follows that

$$E_F\Phi(\xi_\alpha^*(\theta)-U_0+U_1) = E_F\Phi(\xi_\alpha^*(\theta)-U_0) + O(N^{-1-\beta}), \tag{6.26}$$

and hence U_1 may be omitted in (6.25) because its contribution is of negligible order. Retracing our steps back to (6.16) we conclude that the same must be true there, so that

$$\pi_{Pe}(F,\theta) = P(T^* \ge \xi_\alpha^* + O(N^{-1-\beta})) + O(N^{-1-\beta}). \tag{6.27}$$

The remainder of the proof parallels that of theorem 5.2 in ABZ (1976). Let $\tilde{T}$ be Student's statistic as defined in (6.3). The inequality $T^* \ge a$ is algebraically equivalent to $\tilde{T} \ge a\{(N-2)/(N-a^2)\}^{\frac{1}{2}}$ on the set where $\sum(X_i-X_\cdot)^2 \ne 0$ and provided that $a^2 < N$. Since $\sum(X_i-X_\cdot)^2 \ne 0$ on E_1 for sufficiently large N and $\varepsilon' \le \alpha \le 1-\varepsilon'$, this implies that

$$\pi_{Pe}(F,\theta) = P\left(\tilde{T} \ge \xi_\alpha^* + \frac{u_\alpha^3-2u_\alpha}{2N} + O(N^{-1-\beta})\right) + O(N^{-1-\beta}). \tag{6.28}$$

In the same way as in the proof of theorem 5.2 in ABZ (1976) we show that

$$\sup_t P(t \le \tilde{T} \le t + O(N^{-1-\beta})) = O(N^{-1-\beta}) \tag{6.29}$$

and hence

$$\pi_{Pe}(F,\theta) = P\left(\tilde{T} \ge \xi_\alpha^* + \frac{u_\alpha^3-2u_\alpha}{2N}\right) + O(N^{-1-\beta}). \tag{6.30}$$

Now ξ_α^* depends only on N, λ, α and F but not on θ, and arguing as in the proof of theorem 5.2 in ABZ (1976) we find that this together with $\pi_{Pe}(F,0) = \alpha$ ensures that

$$t_\alpha = \xi_\alpha^* + \frac{u_\alpha^3 - 2u_\alpha}{2N} + O(N^{-1-\beta}), \tag{6.31}$$

with t_α defined as in (6.3). Combination of (6.29), (6.30) and (6.31) completes the proof. □

Although we have conducted the proof in such a way as to avoid actually establishing expansions for $\pi_{Pe}(F,\theta)$ and $\pi_{St}(F,\theta)$, the excursion from (6.16) to (6.26) and back has, in fact, brought us rather close to obtaining such expansions. Suppose that the conditions of theorem 6.1 are satisfied but drop the assumption $\int x dF(x) = 0$ that was made in the proof merely for convenience. Define

$$\hat{\eta} = \frac{\{\lambda(1-\lambda)N\}^{\frac{1}{2}}\theta}{\sigma(X_1)}, \tag{6.32}$$

$$\kappa_3(F) = \frac{E(X_1-EX_1)^3}{\sigma^3(X_1)}, \qquad \kappa_4(F) = \frac{E(X_1-EX_1)^4}{\sigma^4(X_1)} - 3, \tag{6.33}$$

where all moments are computed under F since only X_1 is involved. A relatively straightforward computation starting with (6.25) and (6.26) yields

$$\begin{aligned}\pi_{Pe}(F,\theta) = 1 - \Phi(u_\alpha-\hat{\eta}) + \frac{\hat{\eta}\phi(u_\alpha-\hat{\eta})}{72}\Bigg[&\frac{12(1-2\lambda)\kappa_3(F)}{\{\lambda(1-\lambda)N\}^{\frac{1}{2}}}(\hat{\eta}-2u_\alpha) \\ &+ \frac{(1-2\lambda)^2\kappa_3^2(F)}{\lambda(1-\lambda)N}(-\hat{\eta}^4+5u_\alpha\hat{\eta}^3-8u_\alpha^2\hat{\eta}^2+4u_\alpha^3\hat{\eta}+8\hat{\eta}^2-24u_\alpha\hat{\eta}+20u_\alpha^2-10) \\ &+ \frac{3\kappa_4(F)}{\lambda(1-\lambda)N}\{-(1-3\lambda+3\lambda^2)(\hat{\eta}^2-3)+3(1-5\lambda+5\lambda^2)u_\alpha\hat{\eta}-3(1-6\lambda+6\lambda^2)u_\alpha^2\} \\ &- \frac{18u_\alpha^2}{N}\Bigg] + O(N^{-1-\beta}),\end{aligned} \tag{6.34}$$

where β is given by (6.6) and the O symbol is uniform for fixed c,C,D,ε,ε' and δ. Theorem 6.1 ensures that the same expansion is valid for $\pi_{St}(F,\theta)$.

The case where F is normal is perhaps of most interest because both tests are then asymptotically efficient. Since Φ satisfies the stronger regularity conditions needed to replace β by $\frac{1}{2}$ we find in this case

$$\pi_{Pe}(\Phi,\theta) = \pi_{st}(\Phi,\theta) + O(N^{-3/2}) \tag{6.35}$$

$$= 1 - \Phi(u_\alpha - \eta^*) - \frac{u_\alpha^2 \eta^* \phi(u_\alpha - \eta^*)}{4N} + O(N^{-3/2}),$$

where $\eta^* = \{\lambda(1-\lambda)N\}^{\frac{1}{2}}\theta$.

7. DEFICIENCIES OF DISTRIBUTIONFREE TESTS

In analogy to the one-sample case we want to compare the distributionfree tests discussed so far to the best parametric tests for the two-sample problem when the hypothesis and the alternative are both simple. The situation is more complicated than in the one-sample case because of the shift invariance of the distributionfree tests involved. Let $X_1,\dots,X_N$ be independent and let (F,G) denote the hypothesis that $X_1,\dots,X_m$ have common d.f. F and $X_{m+1},\dots,X_N$ have common d.f. G. For fixed F and θ and varying $\Delta \in R^1$, consider the simple hypothesis H_F and the simple alternative $K_{F,\theta,\Delta}$, where

$$H_F = (F,F), \qquad K_{F,\theta,\Delta} = (F(\cdot+\Delta\theta),F(\cdot-(1-\Delta)\theta)).$$

The shift invariance of the distributionfree tests ensures that their power against $K_{F,\theta,\Delta}$ is independent of Δ, so that it was sufficient to consider only alternatives with $\Delta = 0$ in the preceding sections. Note that the form of the locally most powerful rank test against $K_{F,\theta,\Delta}$ is also independent of Δ. However, the envelope power $\pi^+(F,\theta,\Delta)$, i.e. the power of the most powerful level α test of H_F against $K_{F,\theta,\Delta}$, does depend on Δ and the "right" Δ against which comparisons should be made is thus the value Δ_0 that minimizes the envelope power. It is given to first order by $\Delta_0 \sim \lambda$. For values of Δ whose asymptote is different there is not even an asymptotically efficient shift invariant test, so that the deficiency of a shift invariant test with respect to the best test is not of much interest in this case. Of course we shall have to provide a more precise asymptotic evaluation of Δ_0 because we are concerned with second order terms.

Suppose that F is a fixed d.f. with density f that is positive and five times differentiable on R^1. The most powerful level α test for H_F against $K_{F,\theta,\Delta}$ rejects H_F for large values of the statistic

$$S_{\theta,\Delta} = \sum_{i=1}^{m} \log \frac{f(X_i+\Delta\theta)}{f(X_i)} + \sum_{i=m+1}^{N} \log \frac{f(X_i-(1-\Delta)\theta)}{f(X_i)} .$$

This statistic is a sum of independent r.v.'s and we can therefore obtain an Edgeworth expansion for its d.f. under H_F and under $K_{F,\theta,\Delta}$ and hence for the power $\pi^+(F,\theta,\Delta)$ by proceeding in the classical manner and expanding the

cumulants of the statistic. In this expansion for $\pi^+(F,\theta,\Delta)$ we minimize with respect to Δ. We shall give each of these expansions but we omit the tedious computations.

Define Ψ_i by (4.15) for i = 1,...,5, and take

$$\tilde{\pi}^+(F,\theta,\Delta) = 1 - \Phi(u_\alpha - \tilde{\eta}) \tag{7.1}$$

$$- \frac{\tilde{\eta}\phi(u_\alpha-\tilde{\eta})}{288}\left[\frac{24}{N^{\frac{1}{2}}}\frac{\tau_3\int\Psi_1^3(t)dt}{\{\tau_2\int\Psi_1^2(t)dt\}^{3/2}}(-2u_\alpha+\tilde{\eta})\right.$$

$$+ \frac{4}{N}\frac{\tau_4\int\Psi_1^4(t)dt}{\{\tau_2\int\Psi_1^2(t)dt\}^2}\{-3(u_\alpha^2-1)+3\tilde{\eta}u_\alpha-2\tilde{\eta}^2\} + \frac{12}{N}\frac{\tau_4\int\Psi_2^2(t)dt}{\{\tau_2\int\Psi_1^2(t)dt\}^2}\tilde{\eta}^2$$

$$+ \frac{1}{N}\frac{\{\tau_3\int\Psi_1^3(t)dt\}^2}{\{\tau_2\int\Psi_1^2(t)dt\}^3}\{8(2u_\alpha^2-1)-4\tilde{\eta}(u_\alpha^3+3u_\alpha)+\tilde{\eta}^2(8u_\alpha^2+1)-5\tilde{\eta}^3u_\alpha+\tilde{\eta}^4\}$$

$$\left. - \frac{36}{N}\frac{\tau_4}{\tau_2^2}(-u_\alpha^2+1+\tilde{\eta}u_\alpha)\right],$$

where

$$\tilde{\eta} = \left[N\{(1-\lambda)\Delta^2 + \lambda(1-\Delta)^2\}\int\Psi_1^2(t)dt\right]^{\frac{1}{2}}\theta, \tag{7.2}$$

$$\tau_k = (1-\lambda)\Delta^k + \lambda(\Delta-1)^k, \qquad k = 2,3,4. \tag{7.3}$$

LEMMA 7.1. *Let* F *satisfy* (5.6) *for* $m_i = 5/i$, i = 1,...,5, *and suppose that positive numbers* D, D', ε *and* ε' *exist such that* $0 \le \theta \le DN^{-\frac{1}{2}}$, $|\Delta\theta| \le D'N^{-\frac{1}{2}}$, $\varepsilon \le \lambda \le 1-\varepsilon$ *and* $\varepsilon' \le \alpha \le 1-\varepsilon'$. *Then there exists* B > 0 *depending only on* F, D, D', ε *and* ε' *such that*

$$|\pi^+(F,\theta,\Delta) - \tilde{\pi}^+(F,\theta,\Delta)| \le BN^{-3/2}. \tag{7.4}$$

PROOF. Under the conditions of the lemma we find that under $H_F = (F,F)$

$$P_F\left(\frac{S_{\theta,\Delta}+\frac{1}{2}\tilde{\eta}^2}{\tilde{\eta}} \le x\right) = \Phi(x)$$

$$- \frac{\phi(x)}{288}\left[\frac{24}{N^{\frac{1}{2}}}\frac{\tau_3\int\Psi_1^3(t)dt}{\{\tau_2\int\Psi_1^2(t)dt\}^{3/2}}\{2(x^2-1)-3\tilde{\eta}x+\tilde{\eta}^2\} + \right.$$

$$+\frac{4}{N}\frac{\tau_4\int\psi_1^4(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^2}\{3(x^3-3x)-6\tilde{\eta}(x^2-1)+5\tilde{\eta}^2x-2\tilde{\eta}^3\}$$

$$+\frac{12}{N}\frac{\tau_4\int\psi_2^2(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^2}\{-\tilde{\eta}^2x+\tilde{\eta}^3\}$$

$$+\frac{1}{N}\frac{\{\tau_3\int\psi_1^3(t)dt\}^2}{\{\tau_2\int\psi_1^2(t)dt\}^3}\{4(x^5-10x^3+15x)-12\tilde{\eta}(x^4-6x^2+3)+13\tilde{\eta}^2(x^3-3x)$$

$$-6\tilde{\eta}^3(x^2-1)+\tilde{\eta}^4x\}+\frac{36\tau_4}{N\tau_2^2}\{-(x^3-3x)+2\tilde{\eta}(x^2-1)-\tilde{\eta}^2x\}\Big]+O(N^{-3/2}),$$

whereas under $K_{F,\theta,\Delta}$,

$$P\left(\frac{S_{\theta,\Delta}-\frac{1}{2}\tilde{\eta}^2}{\tilde{\eta}}\le x\right)=\Phi(x)$$

$$-\frac{\phi(x)}{288}\Big[\frac{24}{N^{\frac{1}{2}}}\frac{\tau_3\int\psi_1^3(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^{3/2}}\{2(x^2-1)+3\tilde{\eta}x+\tilde{\eta}^2\}$$

$$+\frac{4}{N}\frac{\tau_4\int\psi_1^4(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^2}\{3(x^3-3x)+6\tilde{\eta}(x^2-1)+5\tilde{\eta}^2x+2\tilde{\eta}^3\}$$

$$+\frac{12}{N}\frac{\tau_4\int\psi_2^2(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^2}\{-\tilde{\eta}^2x-\tilde{\eta}^3\}$$

$$+\frac{1}{N}\frac{\{\tau_3\int\psi_1^3(t)dt\}^2}{\{\tau_2\int\psi_1^2(t)dt\}^3}\{4(x^5-10x^3+15x)+12\tilde{\eta}(x^4-6x^2+3)+13\tilde{\eta}^2(x^3-3x)$$

$$+6\tilde{\eta}^3(x^2-1)+\tilde{\eta}^4x\}+\frac{36\tau_4}{N\tau_2^2}\{-(x^3-3x)-2\tilde{\eta}(x^2-1)-\tilde{\eta}^2x\}\Big]+O(N^{-3/2}).$$

The remainder terms $O(N^{-3/2})$ are uniform for fixed F, D, D', ε and ε'. Together these expansions yield (7.4). □

For large values of $|\Delta\theta N^{\frac{1}{2}}|$, both $\pi^+(F,\theta,\Delta)$ and $\tilde{\pi}^+(F,\theta,\Delta)$ will come close to 1 as $N\to\infty$. It follows that an asymptotic expansion for the value Δ_0 that minimizes $\pi^+(F,\theta,\Delta)$ may be obtained by minimizing $\tilde{\pi}^+(F,\theta,\Delta)$ instead.

This yields

$$\Delta_0 = \lambda + \frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}}{4N^{\frac{1}{2}}} \frac{\int \Psi_1^3(t)dt}{\{\int \Psi_1^2(t)dt\}^{3/2}} (-2u_\alpha + \eta^*) + O(N^{-1}) \tag{7.5}$$

with η^* as in (5.29). Since the derivative of $\tilde{\eta}$ with respect to Δ vanishes at $\Delta = \lambda$, (7.5) is sufficient to determine $\tilde{\eta}$ and $\tilde{\pi}^+$ for $\Delta = \Delta_0$ up to a remainder $O(N^{-3/2})$. Noting that indeed $|\Delta_0\theta| = O(N^{-\frac{1}{2}})$, we substitute (7.5) for Δ in (7.1)-(7.3) and neglecting terms that are $O(N^{-3/2})$, we find that $\tilde{\pi}^+(F,\theta,\Delta_0)$ reduces to

$$\begin{aligned}
\tilde{\pi}^+(F,\theta) &= 1 - \Phi(u_\alpha - \eta^*) \\
&+ \frac{\eta^* \phi(u_\alpha - \eta^*)}{288} \Bigg[\frac{24(1-2\lambda)}{\{\lambda(1-\lambda)N\}^{\frac{1}{2}}} \frac{\int \Psi_1^3(t)dt}{\{\int \Psi_1^2(t)dt\}^{3/2}} (-2u_\alpha + \eta^*) \\
&+ \frac{4(1-3\lambda+3\lambda^2)}{\lambda(1-\lambda)N} \frac{\int \Psi_1^4(t)dt}{\{\int \Psi_1^2(t)dt\}^2} \{3(u_\alpha^2-1) - 3\eta^* u_\alpha + 2\eta^{*2}\} \\
&- \frac{12(1-3\lambda+3\lambda^2)}{\lambda(1-\lambda)N} \frac{\int \Psi_2^2(t)dt}{\{\int \Psi_1^2(t)dt\}^2} \eta^{*2} - \frac{9}{N} \frac{\{\int \Psi_1^3(t)dt\}^2}{\{\int \Psi_1^2(t)dt\}^3} (2u_\alpha - \eta^*)^2 \\
&+ \frac{(1-2\lambda)^2}{\lambda(1-\lambda)N} \frac{\{\int \Psi_1^3(t)dt\}^2}{\{\int \Psi_1^2(t)dt\}^3} \{-8(2u_\alpha^2-1) + 4\eta^*(u_\alpha^3+3u_\alpha) - \eta^{*2}(8u_\alpha^2+1) \\
&+ 5\eta^{*3} u_\alpha - \eta^{*4}\} + \frac{36(1-3\lambda+3\lambda^2)}{\lambda(1-\lambda)N} \{-(u_\alpha^2-1) + \eta^* u_\alpha\} \Bigg]
\end{aligned} \tag{7.6}$$

with η^* as in (5.29). Summarizing we have

LEMMA 7.2. *Let* F *satisfy* (5.6) *for* $m_i = 5/i$, $i = 1,\ldots,5$, *and suppose that positive numbers* D, ε *and* ε' *exist such that* $0 \le \theta \le DN^{-\frac{1}{2}}$, $\varepsilon \le \lambda \le 1-\varepsilon$ *and* $\varepsilon' \le \alpha \le 1-\varepsilon'$. *Then there exists* $B > 0$ *depending only on* F, D, ε *and* ε' *such that*

$$|\pi^+(F,\theta,\Delta_0) - \tilde{\pi}^+(F,\theta)| \le BN^{-3/2}. \tag{7.7}$$

For the same testing problem theorem 5.2 provides an expansion for the power $\pi(F,\theta)$ of the locally most powerful rank test. Together, theorem 5.2

and lemma 7.2 enable us to find an asymptotic expression for the deficiency d_N of the locally most powerful rank test with respect to the most powerful parametric test for H_F against $K_{F,\theta;\Delta_0}$. To ensure that F satisfies the assumptions of both theorem 5.2 and lemma 7.2, we require that $F \in \mathcal{F}_1$, where

DEFINITION 7.1. $\mathcal{F}_1$ is the class of d.f.'s F on R^1 with positive and five times differentiable densities f and such that (5.6) is satisfied for $i = 1,\ldots,5$ with $m_1 = 6$, $m_2 = 3$, $m_3 = 5/3$, $m_4 = 5/4$, $m_5 = 1$, and such that (5.7) holds.

Furthermore, we define

$$
\begin{aligned}
(7.8)\qquad \tilde{d}_{N,0} = \frac{1}{48}\Bigg[& \frac{4\int\psi_1^4(t)dt}{\{\int\psi_1^2(t)dt\}^2}\{3(u_\alpha^2-1) - 2\eta^* u_\alpha\} \\
& - \frac{4(1-3\lambda+3\lambda^2)}{\lambda(1-\lambda)}\left\{\frac{\int\psi_1^4(t)dt - 3\int\psi_2^2(t)dt}{\{\int\psi_1^2(t)dt\}^2} + 3\right\}\eta^{*2} \\
& - \frac{3\{\int\psi_1^3(t)dt\}^2}{\{\int\psi_1^2(t)dt\}^3}(2u_\alpha-\eta^*)^2 - \frac{3(1-2\lambda)^2}{\lambda(1-\lambda)}\frac{\{\int\psi_1^3(t)dt\}^2}{\{\int\psi_1^2(t)dt\}^3}\eta^{*2} \\
& - 12\{u_\alpha^2 + 3 - 2\eta^* u_\alpha\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{d}_{N,1} &= \tilde{d}_{N,0} + \frac{1}{\int\psi_1^2(t)dt}\sum_{j=1}^{N}\sigma^2(\psi_1(U_{j:N})), \\
\tilde{d}_{N,2} &= \tilde{d}_{N,0} + \frac{1}{\int\psi_1^2(t)dt}\int_{N^{-1}}^{1-N^{-1}}(\psi_1'(t))^2 t(1-t)dt, \\
\tilde{d}_{N,3} &= \tilde{d}_{N,0} + \frac{1}{\int\psi_1^2(t)dt}\sum_{j=1}^{N}E\left(\psi_1(U_{j:N}) - \psi_1\left(\frac{j}{N+1}\right)\right)^2,
\end{aligned}
$$

where ψ_i and η^* are given by (4.15) and (5.29) and $U_{j:N}$ is the j-th order statistic of a sample of size N from the uniform distribution on (0,1).

THEOREM 7.1. *Let d_N be the deficiency of the locally most powerful rank test with respect to the most powerful test for testing H_F against K_{F,θ,Δ_0} on the basis of $X_1,\ldots,X_N$ and at level α. Suppose that $F \in \mathcal{F}_1$,*

$cN^{-\frac{1}{2}} \le \theta \le CN^{-\frac{1}{2}}$, $\varepsilon \le \lambda \le 1-\varepsilon$ *and* $\varepsilon' \le \alpha \le 1-\varepsilon'$ *for positive* c, C, ε *and* ε'. *Then, for every fixed* F, c, C, ε *and* ε', *there exist positive numbers* B, δ_1, δ_2,... *with* $\lim_{N\to\infty} \delta_N = 0$ *such that*

$$|d_N - \tilde{d}_{N,1}| \le \delta_N, \tag{7.9}$$

$$|d_N - \tilde{d}_{N,2}| \le \delta_N + BN^{-\frac{1}{2}} \int_{N^{-1}}^{1-N^{-1}} (\psi_1'(t))^2 \{t(1-t)\}^{\frac{1}{2}}\, dt. \tag{7.10}$$

If in the above the locally most powerful rank test is replaced by the rank test with the corresponding approximate scores $a_j = -\psi_1(j/(N+1))$ *then*

$$|d_N - \tilde{d}_{N,3}| \le \delta_N \tag{7.11}$$

and (7.10) *continues to hold.*

PROOF. Let us first consider the locally most powerful rank test and show that the expansions (5.35) and (7.7) yield (7.9). The conditions of the theorem ensure that η^*, $\{\lambda(1-\lambda)\}^{-1}$ and u_α are bounded. As $F_1 \subset F$, (5.18) holds and the reasoning leading up to (5.46) gives

$$N^{-1} \sum_{j=1}^{N} \sigma^2(\psi_1(U_{j:N})) \le N^{-1} \sum_{j=1}^{N} E\left\{\psi_1(U_{j:N}) - \psi_1\left(\frac{j}{N+1}\right)\right\}^2 = o(N^{-2/3}). \tag{7.12}$$

In view of these remarks we find from (5.35), (5.32) and (5.31) that the power $\pi(F,\theta)$ of the locally most powerful rank test satisfies

$$\pi(F,\theta) = 1 - \Phi(u_\alpha - \eta^*) + \frac{\eta^* \phi(u_\alpha - \eta^*)}{12} \frac{(1-2\lambda)}{\{\lambda(1-\lambda)N\}^{\frac{1}{2}}} \frac{\int \psi_1^3(t)\,dt}{\{\int \psi_1^2(t)\,dt\}^{3/2}} (-2u_\alpha + \eta^*) + o(N^{-2/3}). \tag{7.13}$$

From lemma 7.2 and (7.6) it is clear that $\pi^+(F,\theta,\Delta_0)$ also equals the right-hand side of (7.13). Since d_N is obtained by replacing N and η^* by $(N+d_N)$ and $\eta^*(1+d_N N^{-1})^{\frac{1}{2}}$ in $\pi(F,\theta)$ and equating the result to $\pi^+(F,\theta,\Delta_0)$, and since η^* is bounded away from zero, we find that $d_N = o(N^{1/3})$.

Having obtained this crude bound for d_N we study the effect of the substitution of $(N+d_N)$ and $\eta^*(1+d_N N^{-1})^{\frac{1}{2}}$ for N and η^* a bit more carefully. The effect on $\pi_0^*(F,\theta)$ as given in (5.31) is obviously the addition of a term

$$(7.14) \qquad \frac{\eta^*\phi(u_\alpha-\eta^*)}{2N} d_N + o(N^{-7/6}) ;$$

to prove that this remains true for $\pi_1^*(F,\theta)$ in (5.32) it is clearly sufficient to show that

$$(7.15) \qquad \frac{1}{N}\sum_{j=1}^{N} \sigma^2(\Psi_1(U_{j:N})) = \frac{1}{N+1}\sum_{j=1}^{N+1} \sigma^2(\Psi_1(U_{j:N+1})) + o(N^{-5/3}).$$

Once this has been established, (5.35) and (7.7) imply that an expansion for d_N may be obtained by equating (7.14) to $\tilde{\pi}^+(F,\theta) - \pi_1^*(F,\theta) + o(N^{-1})$ and an easy computation yields (7.9).

To prove (7.15), we let $b_{j,N}$ denote the density of $U_{j:N}$ and we note the well-known recurrence relation $(N+1)b_{j,N} = jb_{j+1,N+1} + (N-j+1)b_{j,N+1}$. We have

$$\sigma^2(\Psi_1(U_{j:N})) = \frac{j}{N+1} E\{\Psi_1(U_{j+1:N+1}) - E\Psi_1(U_{j:N})\}^2$$

$$+ \frac{N-j+1}{N+1} E\{\Psi_1(U_{j:N+1}) - E\Psi_1(U_{j:N})\}^2$$

$$= \frac{j}{N+1} \sigma^2(\Psi_1(U_{j+1:N+1})) + \frac{N-j+1}{N+1} \sigma^2(\Psi_1(U_{j:N+1}))$$

$$+ \frac{j(N-j+1)}{(N+1)^2} \{E[\Psi_1(U_{j+1:N+1}) - \Psi_1(U_{j:N+1})]\}^2.$$

Summation on j gives

$$(7.16) \qquad \frac{1}{N}\sum_{j=1}^{N} \sigma^2(\Psi_1(U_{j:N}))$$

$$= \frac{1}{N+1}\sum_{j=1}^{N+1} \sigma^2(\Psi_1(U_{j:N+1})) + \sum_{j=1}^{N} \frac{j(N-j+1)}{N(N+1)^2}\{E[\Psi_1(U_{j+1:N+1}) - \Psi_1(U_{j:N+1})]\}^2.$$

By Fubini's theorem and (5.18)

$$|E[\psi_1(U_{j+1:N+1}) - \psi_1(U_{j:N+1})]| \le E \int_{U_{j:N+1}}^{U_{j+1:N+1}} |\psi_1'(t)|\,dt$$

$$= \int_0^1 |\psi_1'(t)|\, P(U_{j:N+1} \le t < U_{j+1:N+1})\,dt = \binom{N+1}{j} \int_0^1 |\psi_1'(t)|\, t^j (1-t)^{N+1-j}\,dt$$

$$\le M\left(\frac{j}{N+1}\right) \frac{(N+1)^{4/3}}{\{j(N+1-j)\}^{7/6}},$$

where M is a bounded function on $(0,1)$ with $\lim_{t\to 0,1} M(t) = 0$. Hence

$$\sum_{j=1}^{N} \frac{j(N-j+1)}{N(N+1)^2} \{E[\psi_1(U_{j+1:N+1}) - \psi_1(U_{j:N+1})]\}^2$$

$$= o\left(N^{-2} \int_{N^{-1}}^{1-N^{-1}} \{t(1-t)\}^{-4/3}\,dt\right) = o(N^{-5/3}).$$

Together with (7.16), this proves (7.15) and establishes expansion (7.9).

For the rank test based on the approximate scores the proof that (5.39) and (7.7) yield expansion (7.11) proceeds in the same way as above, the only difference being that instead of (7.15) we now show that

$$\frac{1}{N} \sum_{j=1}^{N} E\left(\psi_1(U_{j:N}) - \psi_1\left(\frac{j}{N+1}\right)\right)^2 \tag{7.17}$$

$$= \frac{1}{N+1} \sum_{j=1}^{N+1} E\left(\psi_1(U_{j:N+1}) - \psi_1\left(\frac{j}{N+2}\right)\right)^2 + o(N^{-5/3}).$$

Using the recurrence relation for $b_{j,N}$ again, we find after some arithmetic

$$\frac{1}{N} \sum_{j=1}^{N} E\left(\psi_1(U_{j:N}) - \psi_1\left(\frac{j}{N+1}\right)\right)^2 \tag{7.18}$$

$$= \frac{1}{N+1} \sum_{j=1}^{N+1} E\left(\psi_1(U_{j:N+1}) - \psi_1\left(\frac{j}{N+2}\right)\right)^2 +$$

$$+\frac{2}{N+1}\sum_{j=1}^{N+1}\left\{\psi_1\left(\frac{j}{N+2}\right)-\frac{j-1}{N}\,\psi_1\left(\frac{j-1}{N+1}\right)-\frac{N-j+1}{N}\,\psi_1\left(\frac{j}{N+1}\right)\right\}E\left(\psi_1(U_{j:N+1})-\psi_1\left(\frac{j}{N+2}\right)\right)$$

$$+\frac{1}{N+1}\sum_{j=1}^{N+1}\left\{\frac{j-1}{N}\left(\psi_1\left(\frac{j}{N+2}\right)-\psi_1\left(\frac{j-1}{N+1}\right)\right)^2+\frac{N-j+1}{N}\left(\psi_1\left(\frac{j}{N+2}\right)-\psi_1\left(\frac{j}{N+1}\right)\right)^2\right\}.$$

Now (5.18) ensures that

$$\left|\psi_1\left(\frac{j}{N+2}\right)-\psi_1\left(\frac{j-1}{N+1}\right)\right| \le \tilde{M}\left(\frac{j}{N+2}\right)\frac{(N-j+2)}{(N+2)^2}\left\{\frac{j(N-j+2)}{(N+2)^2}\right\}^{-7/6} \quad \text{for } j = 2,\ldots,N+1,$$

$$\left|\psi_1\left(\frac{j}{N+2}\right)-\psi_1\left(\frac{j}{N+1}\right)\right| \le \tilde{M}\left(\frac{j}{N+2}\right)\frac{j}{(N+2)^2}\left\{\frac{j(N-j+2)}{(N+2)^2}\right\}^{-7/6} \quad \text{for } j = 1,\ldots,N,$$

where $\tilde{M}$ is a bounded function on (0,1) with $\lim_{t\to 0,1}\tilde{M}(t) = 0$. Similarly, (5.7), lemma A2.3 in ABZ (1976) and (5.18) imply that

$$\left|E\psi_1(U_{j:N+1})-\psi_1\left(\frac{j}{N+2}\right)\right| \le \tilde{M}_2\left(\frac{j}{N+2}\right)N^{-1}\left\{\frac{j(N-j+2)}{(N+2)^2}\right\}^{-7/6} \quad \text{for } j = 1,\ldots,N+1.$$

It follows that both the second and third terms on the right in (7.18) are

$$o\left(N^{-2}\int_{N^{-1}}^{1-N^{-1}}\{t(1-t)\}^{-4/3}\,dt\right) = o(N^{-5/3}),$$

which proves (7.17) and therefore (7.11).

Finally, the validity of expansion (7.10) for exact as well as approximate scores is a simple consequence of (7.9) and (7.11) and the fact that theorem 5.2 clearly implies that both $\sum\sigma^2(\psi_1(U_{j:N}))$ and $\sum E(\psi_1(U_{j:N})-\psi_1(j/(N+1)))^2$ equal

$$\int_{N^{-1}}^{1-N^{-1}}(\psi_1'(t))^2t(1-t)dt + o(1) + O\left(N^{-\frac{1}{2}}\int_{N^{-1}}^{1-N^{-1}}(\psi_1'(t))^2\{t(1-t)\}^{\frac{1}{2}}dt\right).$$

This completes the proof of the theorem. □

Like theorems 5.1 and 5.2, theorem 7.1 presents us with a choice between an expansion with remainder $o(1)$ and one which is more explicit but may have a remainder of larger order under the conditions of the theorem. If $\Psi_1'(t) = o(\{t(1-t)\}^{-1})$ for $t \to 0,1$, then $d_N = \tilde{d}_{N,2} + o(1)$ for exact as well as approximate scores and expansion (7.10) is obviously preferable. This appears to be the most common case. However, if $\Psi_1'(t)$ is of exact order $\{t(1-t)\}^{-1}$, then (7.10) yields only

$$d_N = \frac{\int_{N^{-1}}^{1-N^{-1}} (\Psi_1'(t))^2 t(1-t)dt}{\int_0^1 \Psi_1^2(t)dt} + O(1) = O(\log N).$$

Finally, if $\Psi_1'(t) \sim \{t(1-t)\}^{-1-\delta}$ for $t \to 0,1$ and some $0 < \delta < 1/6$, then (7.10) reduces to $d_N = O(N^{2\delta})$.

In general, all we can say under the conditions of theorem 7.1 is that

$$(7.19) \qquad d_N = \frac{\sum \sigma^2(\Psi_1(U_{j:N}))}{\int \Psi_1^2(t)dt} + O(1) = O\left(\int_{N^{-1}}^{1-N^{-1}} (\Psi_1'(t))^2 t(1-t)dt \right) = o(N^{1/3})$$

for exact scores, and that

$$(7.20) \qquad d_N = \frac{\sum E\left(\Psi_1(U_{j:N}) - \Psi_1\left(\frac{j}{N+1}\right)\right)^2}{\int \Psi_1^2(t)dt} + O(1)$$

$$= O\left(\int_{N^{-1}}^{1-N^{-1}} (\Psi_1'(t))^2 t(1-t)dt \right) = o(N^{1/3})$$

for approximate scores. Even this result, however, is rather surprising because one might have expected these deficiencies to be of the order $N^{\frac{1}{2}}$. The reason that they are of smaller order than $N^{\frac{1}{2}}$ is of course that the power expansions for the rank tests in theorem 5.2 and for the most powerful test in lemma 7.2 agree not only in their leading terms of order 1 but also in their second order terms of order $N^{-\frac{1}{2}}$. It is only in the third order terms that differences begin to show up. Borrowing a phrase from PFANZAGL (1977) who noted the same phenomenon in the parametric one-sample

problem, first order efficiency apparently implies second order efficiency in the cases considered. Note that results very similar to (7.19) and (7.20) were obtained for one-sample rank tests in ABZ (1976). In that case, however, there is no cause for surprise because certain symmetries that are present in the nonparametric one-sample problem ensure that there is no term of order $N^{-\frac{1}{2}}$ in any of the power expansions. Finally we should perhaps point out that in the present two-sample case, the fact that we have evaluated the envelope power for Δ_0 as given in (7.5) instead of for the conventional choice $\Delta = \lambda$, is of no consequence for these considerations. For $\Delta = \lambda$ the term involving $(2u_\alpha - \eta^*)^2$ should simply be omitted from (7.6) and (7.8) and this doesn't influence the qualitative behavior of $\tilde{\pi}^+$ or $\tilde{d}_{N,i}$.

To provide some examples of theorem 7.1 we compute the expansion (7.10) for the special case where F is the logistic d.f. $\Lambda(x) = (1+e^{-x})^{-1}$ or the normal d.f. Φ. The computations resemble those at the end of section 5. Suppose that $c \le \theta N^{\frac{1}{2}} \le C$, $\varepsilon \le \lambda \le 1-\varepsilon$ and $\varepsilon' \le \alpha \le 1-\varepsilon'$ for positive c, C, ε and ε'. As both examples concern symmetric distributions for which $\int \psi_1^3(t)dt = 0$, the second order term in (7.5) vanishes so that we may take $\Delta_0 = \lambda$ in both cases. For $F = \Lambda$ we are therefore concerned with the problem of testing the hypothesis (Λ,Λ) against the alternative $(\Lambda(\cdot+\lambda\theta),\Lambda(\cdot-(1-\lambda)\theta))$ and d_N denotes the deficiency of Wilcoxon's two-sample test with respect to the most powerful test for this problem. We find

$$(7.21) \qquad d_N = \frac{1}{20}\left[4u_\alpha^2 + 16 + 4\eta^* u_\alpha + \frac{1-3\lambda+3\lambda^2}{\lambda(1-\lambda)}\,\eta^{*2}\right] + o(1)$$

with $\eta^* = \{\lambda(1-\lambda)N/3\}^{\frac{1}{2}}\theta$. In this example d_N remains bounded as $N \to \infty$.

In the second example we consider the testing problem (Φ,Φ) versus $(\Phi(\cdot+\lambda\vartheta),\Phi(\cdot-(1-\lambda)\theta))$. Now d_N is the deficiency of the two-sample normal scores test (or van der Waerden's two-sample test) with respect to the most powerful test based on the difference of the sample means. We obtain

$$(7.22) \qquad d_N = \log\log N + \tfrac{1}{2}(u_\alpha^2 - 3) + \log 2 + \gamma + o(1),$$

where γ denotes Euler's constant (cf. (5.50)). Now $d_N \sim \log\log N \to \infty$ as $N \to \infty$. Note that there is no dependence on θ or λ in this expansion.

So far in this section we have compared distributionfree tests to the most powerful test for a simple hypothesis against a simple alternative. However, all distributionfree tests occurring in this paper - rank tests as well as the permutation test discussed in section 6 - are invariant under changes of location and scale. It would therefore be more realistic to compare these tests to the uniformly most powerful location and scale invariant test, if such a test exists. For the two-sample normal location problem Student's test answers this description and its power would therefore be a more suitable basis for comparison than the envelope power. For the problem of testing (Φ,Φ) against $(\Phi(\cdot+\lambda\theta),\Phi(\cdot-(1-\lambda)\theta))$ the power of the most powerful test equals $1-\Phi(u_\alpha-\eta^*)$ with $\eta^* = \{\lambda(1-\lambda)N\}^{\frac{1}{2}}\theta$. Assuming again that $c \leq \theta N^{\frac{1}{2}} \leq C$, $\varepsilon \leq \lambda \leq 1-\varepsilon$ and $\varepsilon' \leq \alpha \leq 1-\varepsilon'$ for positive c, C, ε and ε', the power of Student's two-sample test is given by (6.35) and its deficiency with respect to the most powerful test is therefore equal to $\frac{1}{2}u_\alpha^2 + o(1)$. It follows from (7.22) that the deficiency of the two-sample normal scores test (or van der Waerden's two-sample test) with respect to Student's two-sample test for the normal location problem is given by

$$(7.23) \qquad d_N = \log\log N - \frac{3}{2} + \log 2 + \gamma + o(1),$$

where now the expansion doesn't even depend on α. Since both tests are location invariant, (7.23) also denotes the deficiency for testing (Φ,Φ) against $(\Phi,\Phi(\cdot-\theta))$.

We conclude this section by comparing the permutation test discussed in section 6 to Student's test. Theorem 7.2 is an immediate consequence of theorem 6.1, expansion (6.34) and (6.8).

THEOREM 7.2. *Suppose that positive numbers* $c,c',C,D,\varepsilon,\varepsilon',\delta$ *and* $r > 8$ *exist such that the conditions of theorem* 6.1 *are satisfied and that* $\theta \geq c'N^{-\frac{1}{2}}$. *Let* d_N *denote the deficiency of the permutation test based on the sample means with respect to Student's two-sample test for testing* (F,F) *against* $(F,F(\cdot-\theta))$ *on the basis of* $X_1,\ldots,X_N$ *and at level* α. *Then there exist* $B > 0$ *depending only on* $c,c',C,D,\varepsilon,\varepsilon'$ *and* δ, *and* $\beta > 0$ *depending only on* r *such that*

$$(7.24) \qquad d_N \leq BN^{-\beta}.$$

The case $F = \Phi$ is of course of most interest because then the theorem asserts that for the normal location problem there exists a distributionfree test whose deficiency with respect to the best location and scale invariant test tends to zero. We note that the remark at the end of section 6 implies that in this case (7.24) may be replaced by $d_N \leq BN^{-\frac{1}{2}}$. For $F \neq \Phi$ the theorem merely shows how closely the permutation test resembles Student's test with the correct significance level for F.

8. EXPANSIONS AND DEFICIENCIES FOR RELATED ESTIMATORS

Let $X_1,\ldots,X_N$ be independent and let (F,G) denote the hypothesis that $X_1,\ldots,X_m$ have common d.f. F and $X_{m+1},\ldots,X_N$ have common d.f. G. Let $T = T(X_1,\ldots,X_N)$ be the rank statistic given by (2.2) and suppose that the scores a_j are nondecreasing in $j = 1,\ldots,N$. Define the statistic M by

$$(8.1)\qquad M(X_1,\ldots,X_N) = \tfrac{1}{2}\sup\{t : T(X_1,\ldots,X_m,X_{m+1}-t,\ldots,X_N-t) > \lambda\textstyle\sum a_j\}$$
$$+ \tfrac{1}{2}\inf\{t : T(X_1,\ldots,X_m,X_{m+1}-t,\ldots,X_N-t) < \lambda\textstyle\sum a_j\}.$$

Under the model $(F,F(\cdot-\mu))$, M was proposed as an estimator of μ by HODGES and LEHMANN [1963]. They showed that the normal approximation to the power of the level $\frac{1}{2}$ test based on T for contiguous location alternatives can be used to establish asymptotic normality of M. In the same way we shall show that a power expansion yields an expansion for the d.f. of $N^{\frac{1}{2}}(M-\mu)$. Note that we donot make the assumption of HODGES and LEHMANN [1963] that the distribution of T under (F,F) is symmetric about $\lambda\sum a_j$, which occurs e.g. when either $\lambda = \frac{1}{2}$ or when the scores are antisymmetric. As a result the power expansion involved will be for the test based on T at level $\bar{\alpha} = \frac{1}{2} + O(N^{-\frac{1}{2}})$ rather than at level $\frac{1}{2}$, but for our deficiency computations this will not make any difference. We shall restrict attention to the case where T is the statistic of the locally most powerful rank test or its approximate scores analogue, so that the a_j will be exact or approximate scores generated by the score function $-\Psi_1$, with Ψ_1 as in (4.15). To ensure that the scores are nondecreasing we require that the density f of F is strongly unimodal, i.e. that $\log f$ is concave.

Let $\bar{F}$ be given by definition 5.1, let $\pi(\alpha,F,\theta)$ denote the power of the level α right-sided test based on T against the alternative $(F,F(\cdot-\theta))$ and define

$$(8.2)\qquad \bar{\alpha} = \tfrac{1}{2} + \frac{(1-2\lambda)}{6\{2\pi\lambda(1-\lambda)N\}^{\frac{1}{2}}}\,\frac{\int\Psi_1^3(t)dt}{\{\int\Psi_1^2(t)dt\}^{3/2}}\,.$$

Furthermore define, with Ψ_i as in (4.15),

$$(8.3)\qquad \bar{L}_0(x) = \Phi(x) - \frac{\phi(x)}{288}\Bigg[\frac{24(1-2\lambda)}{\{\lambda(1-\lambda)N\}^{\frac{1}{2}}}\frac{\int\psi_1^3(t)dt}{\{\int\psi_1^2(t)dt\}^{3/2}}(x^2+2)$$

$$- \frac{4}{\lambda(1-\lambda)N}\frac{\int\psi_1^4(t)dt}{\{\int\psi_1^2(t)dt\}^2}\{5(1-3\lambda+3\lambda^2)x^3 - 3(1-6\lambda+6\lambda^2)x\}$$

$$+ \frac{48(1-3\lambda+3\lambda^2)}{\lambda(1-\lambda)N}\frac{\int\psi_2^2(t)dt}{\{\int\psi_1^2(t)dt\}^2}x^3 + \frac{(1-2\lambda)^2}{\lambda(1-\lambda)N}\frac{\{\int\psi_1^3(t)dt\}^2}{\{\int\psi_1^2(t)dt\}^3}(x^5-4x^3-12x)$$

$$- \frac{36}{\lambda(1-\lambda)N}\{(1-3\lambda+3\lambda^2)x^3 + x\}\Bigg],$$

$$\bar{L}_1(x) = \bar{L}_0(x) - \frac{x\phi(x)}{2N\int\psi_1^2(t)dt}\sum_{j=1}^{N}\sigma^2(\psi_1(U_{j:N})),$$

$$\bar{L}_2(x) = \bar{L}_0(x) - \frac{x\phi(x)}{2N\int\psi_1^2(t)dt}\int_{N^{-1}}^{1-N^{-1}}(\psi_1'(t))^2t(1-t)dt,$$

$$\bar{L}_3(x) = \bar{L}_0(x) - \frac{x\phi(x)}{2N\int\psi_1^2(t)dt}\sum_{j=1}^{N}E\left(\psi_1(U_{j:N}) - \psi_1\left(\frac{j}{N+1}\right)\right)^2.$$

Probabilities under the model $(F,F(\cdot-\mu))$ are denoted by $P_{F,\mu}$.

THEOREM 8.1. *Suppose that* $F \in \mathcal{F}$, *that* f *is strongly unimodal and that either* $a_j = -E\psi_1(U_{j:N})$ *for* $j = 1,\ldots,N$, *or* $a_j = -\psi_1\left(\frac{j}{N+1}\right)$ *for* $j = 1,\ldots,N$. *Let* ε *and* C *be positive numbers and suppose that* $\varepsilon \le \lambda \le 1-\varepsilon$. *Then there exist positive numbers* $B,\delta_1,\delta_2,\ldots$, *with* $\lim_{N\to\infty}\delta_N = 0$, *which depend only on* F, ε *and* C, *such that*

$$(8.4)\qquad \sup_{|\xi|\le C}\left|P_{F,\mu}(N^{\frac{1}{2}}(M-\mu) \le \xi) - \{1 - \pi(\bar{\alpha},F,-\xi N^{-\frac{1}{2}})\}\right| \le \delta_N N^{-1}$$

and such that the following statements hold.

(i) For exact scores $a_j = -E\psi_1(U_{j:N})$,

$$(8.5)\qquad \sup_{|x|\le C}\left|P_{F,\mu}\left(\{\lambda(1-\lambda)N\int\psi_1^2(t)dt\}^{\frac{1}{2}}(M-\mu) \le x\right) - \bar{L}_1(x)\right| \le \delta_N N^{-1},$$

$$(8.6)\qquad \sup_{|x|\le C}\left|P_{F,\mu}\left(\{\lambda(1-\lambda)N\int\psi_1^2(t)dt\}^{\frac{1}{2}}(M-\mu)\le x\right)-\bar{L}_2(x)\right|\le\delta_N N^{-1}$$
$$+\,BN^{-3/2}\int_{N^{-1}}^{1-N^{-1}}(\psi_1'(t))^2\{t(1-t)\}^{\frac{1}{2}}dt;$$

(ii) For approximate scores $a_j=-\psi_1\left(\frac{j}{N+1}\right)$,

$$(8.7)\qquad \sup_{|x|\le C}\left|P_{F,\mu}(\{\lambda(1-\lambda)N\int\psi_1^2(t)dt\}^{\frac{1}{2}}(M-\mu)\le x)-\bar{L}_3(x)\right|\le\delta_N N^{-1}$$

and (8.6) *continues to hold.*

PROOF. In view of (8.1) we have for $\theta=-\xi N^{-\frac{1}{2}}$,

$$P_{F,\mu}(N^{\frac{1}{2}}(M-\mu)\le\xi)=P_{F,\theta}(M\le 0),$$

$$P_{F,\theta}(T<\lambda\textstyle\sum a_j)\le P_{F,\theta}(M\le 0)\le P_{F,\theta}(T\le\lambda\sum a_j).$$

For $\theta=-\xi N^{-\frac{1}{2}}$ and $\varepsilon'\le\alpha\le 1-\varepsilon'$, the conditions of theorem 5.2 are satisfied except, of course, that $\theta<0$ if $\xi>0$. However, the theorem remains valid for $|\theta|\le DN^{-\frac{1}{2}}$; it was formulated for positive θ merely because we were discussing one-sided tests against one-sided alternatives at that point. It follows that $P_{F,\theta}(T=\lambda\sum a_j)=o(N^{-1})$ uniformly for $|\xi|\le C$, so that

$$(8.8)\qquad P_{F,\mu}(N^{\frac{1}{2}}(M-\mu)\le\xi)=P_{F,\theta}(T\le\lambda\textstyle\sum a_j)+o(N^{-1})$$
$$=1-\pi(\alpha,F,-\xi N^{-\frac{1}{2}})+o(N^{-1}),$$

where α is the level of the test that rejects if $T>\lambda\sum a_j$. Noting that $\sum a_j=0$ for exact scores, we find from (5.33) and (5.37) for $x=\eta^*=0$, that $\alpha=\bar{\alpha}+o(N^{-1})$. In view of (5.35) and (5.39) this yields $\pi(\alpha,F,-\xi N^{-\frac{1}{2}})=\pi(\bar{\alpha},F,-\xi N^{-\frac{1}{2}})+o(N^{-1})$ uniformly for $|\xi|\le C$ and together with (8.8) this proves (8.4). The remainder of the theorem follows from (8.8) and expansions (5.33), (5.34), (5.37) and (5.38) with x and η^* replaced by 0 and -x. □

The natural parametric competitor of M as an estimator of μ is of course

the maximum likelihood estimator M'. Under the model $K_{F,\mu,\Delta} = (F(\cdot+\Delta\mu), F(\cdot-(1-\Delta)\mu))$, $M' = M'_\Delta$ is the solution of

$$(8.9) \qquad \Delta \sum_{i=1}^{m} \psi_1(X_i+\Delta M') - (1-\Delta) \sum_{i=m+1}^{N} \psi_1(X_i - (1-\Delta)M') = 0$$

with $\psi_1 = f'/f$ as in (4.1). Note that, in contrast to M, the estimator M'_Δ as well as its distribution under $K_{F,\mu,\Delta}$ depend on Δ.

The d.f. of M'_Δ under $K_{F,\mu,\Delta}$ is connected with the power of the locally most powerful test for $H_F = (F,F)$ against $K_{F,\theta,\Delta}$. For $\theta > 0$, this test rejects H_F for large values of the statistic

$$(8.10) \qquad S_\Delta = \Delta \sum_{i=1}^{m} \psi_1(X_i) - (1-\Delta) \sum_{i=m+1}^{N} \psi_1(X_i).$$

Let $\pi(\alpha,F,\theta,\Delta)$ denote the power against $K_{F,\theta,\Delta}$ of this right-sided test at level α. Suppose that F is a fixed d.f. with density f that is positive and five times differentiable on R^1 and define

$$\begin{aligned}(8.11) \qquad \tilde{\pi}(\alpha,F,\theta,\Delta) = {} & 1 - \Phi(u_\alpha - \tilde{\eta}) - \frac{\tilde{\eta}\phi(u_\alpha-\tilde{\eta})}{288}\Bigg[\frac{24}{N^{\frac{1}{2}}} \frac{\tau_3\int\psi_1^3(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^{3/2}} (-2u_\alpha+\tilde{\eta}) \\ & + \frac{4}{N} \frac{\tau_4\int\psi_1^4(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^2} \{-3(u_\alpha^2-1) + 3\tilde{\eta}u_\alpha - 5\tilde{\eta}^2\} + \frac{48}{N} \frac{\tau_4\int\psi_2^2(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^2} \tilde{\eta}^2 \\ & + \frac{1}{N} \frac{\{\tau_3\int\psi_1^3(t)dt\}^2}{\{\tau_2\int\psi_1^2(t)dt\}^3} \{8(2u_\alpha^2-1) - 4\tilde{\eta}(u_\alpha^3+3u_\alpha) + 8\tilde{\eta}^2(u_\alpha^2-1) - 5\tilde{\eta}^3u_\alpha + \tilde{\eta}^4\} \\ & + \frac{36\tau_4}{N\tau_2^2} \{(u_\alpha^2-1) - \tilde{\eta}u_\alpha - \tilde{\eta}^2\}\Bigg],\end{aligned}$$

where $\tilde{\eta}$ and τ_k are given by (7.2) and (7.3).

LEMMA 8.1. *Let* F *satisfy* (5.6) *for* $m_i = 5/i$, $i = 1,\ldots,5$, *and suppose that positive numbers* D, D', ε *and* ε' *exist such that* $|\theta| \le DN^{-\frac{1}{2}}$, $|\Delta\theta| \le D'N^{-\frac{1}{2}}$, $\varepsilon \le \lambda \le 1-\varepsilon$ *and* $\varepsilon' \le \alpha \le 1-\varepsilon'$. *Then there exists* $B > 0$ *depending only on* F, D, D', ε *and* ε' *such that*

$$(8.12) \qquad |\pi(\alpha,F,\theta,\Delta) - \tilde{\pi}(\alpha,F,\theta,\Delta)| \le BN^{-3/2}.$$

PROOF. The proof proceeds in the same manner as that of lemma 7.1 and again we omit the details. Under the conditions of the lemma we find that under $K_{F,\theta,\Delta}$,

$$(8.13)\qquad P\left(\frac{S_\Delta}{\{N\tau_2\int\psi_1^2(t)dt\}^{\frac{1}{2}}} \le x\right) = \Phi(x-\tilde{\eta})$$

$$- \frac{\phi(x-\tilde{\eta})}{288}\left[\frac{24}{N^{\frac{1}{2}}}\frac{\tau_3\int\psi_1^3(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^{3/2}}\{2(x^2-1)+2\tilde{\eta}x-\tilde{\eta}^2\}\right.$$

$$+ \frac{4}{N}\frac{\tau_4\int\psi_1^4(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^2}\{3(x^3-3x)+3\tilde{\eta}(x^2-1)-3\tilde{\eta}^2x+5\tilde{\eta}^3\}$$

$$- \frac{48}{N}\frac{\tau_4\int\psi_2^2(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^2}\tilde{\eta}^3 + \frac{1}{N}\frac{\{\tau_3\int\psi_1^3(t)dt\}^2}{\{\tau_2\int\psi_1^2(t)dt\}^3}\{4(x^5-10x^3+15x)$$

$$+ 4\tilde{\eta}(x^4-6x^2+3) - 8\tilde{\eta}^2(x^3-3x) - 4\tilde{\eta}^3(x^2-1) + 5\tilde{\eta}^4x - \tilde{\eta}^5\}$$

$$\left.+ \frac{36\tau_4}{N\tau_2^2}\{-(x^3-3x) - \tilde{\eta}(x^2-1) + \tilde{\eta}^2x + \tilde{\eta}^3\}\right] + O(N^{-3/2}).$$

The remainder term is uniformly $O(N^{-3/2})$ for fixed F,D,D',ε and ε'. This expansion yields (8.12). □

Note that the expansions (8.12) and (8.13) are valid also for negative values of θ, but that the right-sided test considered here is not locally most powerful against these alternatives.

If the conditions of lemma 8.1 are fulfilled and if, moreover, f is strongly unimodal so that ψ_1 is nonincreasing, then we can establish the connection between $\pi(\alpha,F,\theta,\Delta)$ and the d.f. of M'_Δ by arguing as in the proof of theorem 8.1. Writing $P_{F,\mu,\Delta}$ for probabilities under $K_{F,\mu,\Delta}$ and taking $\theta = -\xi N^{-\frac{1}{2}}$ we find that

$$P_{F,\theta,\Delta}(S_\Delta<0) \le P_{F,\mu,\Delta}(N^{\frac{1}{2}}(M'_\Delta-\mu) \le \xi) \le P_{F,\theta,\Delta}(S_\Delta\le 0).$$

In view of (8.12) and (8.13) this implies that uniformly for $|\xi|$, $|x| \le C$,

$$(8.14)\qquad P_{F,\mu,\Delta}(N^{\frac{1}{2}}(M'_\Delta-\mu)\le\xi) = 1-\pi(\tilde{\alpha},F,\theta,\Delta)+O(N^{-3/2}),$$

$$(8.15)\qquad \tilde{\alpha} = \tfrac{1}{2}-\frac{1}{6\{2\pi N\}^{\frac{1}{2}}}\frac{\tau_3\int\psi_1^3(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^{3/2}},$$

$$(8.16)\qquad P_{F,\mu,\Delta}(\{\tau_2 N\int\psi_1^2(t)dt\}^{\frac{1}{2}}(M'_\Delta-\mu)\le x) = \Phi(x)$$

$$-\frac{\phi(x)}{288}\Big[-\frac{24}{N^{\frac{1}{2}}}\frac{\tau_3\int\psi_1^3(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^{3/2}}(x^2+2)-\frac{4}{N}\frac{\tau_4\int\psi_1^4(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^{2}}(5x^3-3x)$$

$$+\frac{48}{N}\frac{\tau_4\int\psi_2^2(t)dt}{\{\tau_2\int\psi_1^2(t)dt\}^{2}}x^3+\frac{1}{N}\frac{\{\tau_3\int\psi_1^3(t)dt\}^2}{\{\tau_2\int\psi_1^2(t)dt\}^{3}}(x^5-4x^3-12x)$$

$$-\frac{36\tau_4}{N\tau_2^2}(x^3+x)\Big]+O(N^{-3/2}).$$

We have already remarked that the d.f. of $(M'_\Delta-\mu)$ under $K_{F,\mu,\Delta}$ depends on Δ and thus the same problem arises that we encountered in section 7, viz. to determine the "right" Δ for which M and M' should be compared. It is easy to see from (8.16) that the value $\Delta=\Delta^0$ that is least favorable for M' in the sense that it minimizes (maximizes) $P_{F,\mu,\Delta}(\{N\lambda(1-\lambda)\int\psi_1^2(t)dt\}^{\frac{1}{2}}(M'_\Delta-\mu)\le x)$ for positive (negative) x is given by

$$\Delta^0 = \lambda-\frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}}{4N^{\frac{1}{2}}}\frac{\int\psi_1^3(t)dt}{\{\int\psi_1^2(t)dt\}^{3/2}}\frac{x^2+2}{x}+O(N^{-1}).$$

However, we shall not take $\Delta=\Delta^0$ as a basis for comparing M and M' but we shall simply choose $\Delta=\lambda$ instead. We advance three reasons for doing so. The reader who doesn't find these reasons sufficiently compelling should realize that we are merely granting the maximum likelihood estimator a slight additional advantage.

(i). The second order term of Δ^0 depends on x just as the second order term of Δ_0 in (7.5) depends on θ. This didn't deter us from choosing $\Delta=\Delta_0$ as a basis for comparison in section 7, but we feel the situation is slightly different there. In section 7 we were comparing with envelope power and in

general this means comparing with a different most powerful test for each alternative (θ,Δ). This being so, there seems to be little reason not to choose the least favorable testing problem for each value of θ, i.e. to take $\Delta = \Delta_0$. All we are doing is locating a curve $(\theta,\Delta_0(\theta))$ of least favorable alternatives in the set of all alternatives (θ,Δ) and comparing with envelope power on that curve only.

Our attitude would have been different, however, if in section 7 we would have been comparing with the power of the locally most powerful test rather than with the envelope power. The locally most powerful test is of course independent of θ (c.f. (8.10)) and for every fixed Δ we would therefore be comparing with a single fixed test for all θ. In this case it would still be reasonable to choose $\Delta = \lambda$ which is least favorable to first order, but if $\int\psi_1^3(t)dt \neq 0$, it would seem to be rather extreme to compute the power of the locally most powerful test at each θ for $\Delta = \Delta_0 = \Delta_0(\theta)$ which is least favorable to second order in this case too. After all, for every fixed Δ there would be a single locally most powerful test that does better than that for all values of θ except the one for which $\Delta_0(\theta) = \Delta$. It is precisely for such sets of alternatives (Δ fixed, θ unknown) that the locally most powerful test is designed and it seems unrealistic to assess its performance only for a different one-parameter set of alternatives $(\theta,\Delta_0(\theta))$.

The present problem for the maximum likelihood estimator is of course very similar to the one for the locally most powerful test. Again the choice $\Delta = \Delta^0$ depending on x appears to be rather extreme because for every Δ the d.f. of the maximum likelihood estimator is more concentrated around μ than this choice would indicate at all but at most two points.

(ii). Even though, in general, the distribution of M'_Δ under $K_{F,\mu,\Delta}$ is not symmetric about μ, most reasonable measures of dispersion are built around the distribution of $|M'_\Delta-\mu|$ rather than $(M'_\Delta-\mu)$. It is clear from (8.16) that $P_{F,\mu,\Delta}(\{N\lambda(1-\lambda)\int\psi_1^2(t)dt\}^{\frac{1}{2}}\ |M'_\Delta-\mu| \leq x)$ is minimized by $\Delta = \lambda + \mathcal{O}(N^{-1})$; it is also obvious from (8.16) that it makes no difference for our asymptotic results if we take $\Delta = \lambda$ instead (c.f. the remark following (7.5)). Hence $\Delta = \lambda$ is the "right" choice of Δ for our asymptotic comparison of M and M', provided that the comparison is made on the basis of the distributions of $|M-\mu|$ and $|M'-\mu|$.

(iii). Our final argument is the rather more pedestrian one that any choice of Δ other than $\Delta = \lambda + o(N^{-\frac{1}{2}})$ would to a certain extent destroy the simplicity of the main results in this section. We shall elaborate points (ii) and (iii) after proving theorem 8.3.

We now substitute $\Delta = \lambda$ in (8.14)-(8.16) and find that $\tilde{\alpha}$ reduces to $\bar{\alpha}$ as defined in (8.2) and that the expansion on the right in (8.16) becomes

$$(8.17)\qquad L^{*}(x) = \Phi(x) - \frac{\phi(x)}{288}\Bigg[\frac{24(1-2\lambda)}{\{\lambda(1-\lambda)N\}^{\frac{1}{2}}}\frac{\int\psi_1^3(t)dt}{\{\int\psi_1^2(t)dt\}^{3/2}}(x^2+2)$$

$$- \frac{4(1-3\lambda+3\lambda^2)}{\lambda(1-\lambda)N}\frac{\int\psi_1^4(t)dt}{\{\int\psi_1^2(t)dt\}^2}(5x^3-3x) + \frac{48(1-3\lambda+3\lambda^2)}{\lambda(1-\lambda)N}\frac{\int\psi_2^2(t)dt}{\{\int\psi_1^2(t)dt\}^2}x^3$$

$$+ \frac{(1-2\lambda)^2}{\lambda(1-\lambda)N}\frac{\{\int\psi_1^3(t)dt\}^2}{\{\int\psi_1^2(t)dt\}^3}(x^5-4x^3-12x) - \frac{36(1-3\lambda+3\lambda^2)}{\lambda(1-\lambda)N}(x^3+x)\Bigg].$$

We have proved

THEOREM 8.2. *Suppose that* F *satisfies* (5.6) *for* $m_i = 5/i$, $i = 1,\ldots,5$ *and that* f *is strongly unimodal. Let* ε *and* C *be positive numbers and suppose that* $\varepsilon \le \lambda \le 1-\varepsilon$. *Then there exists* $B > 0$ *depending only on* F, ε *and* C, *such that*

$$(8.18)\qquad \sup_{|\xi|\le C}\left|P_{F,\mu,\lambda}(N^{\frac{1}{2}}(M'_\lambda-\mu) \le \xi) - \{1 - \pi(\bar{\alpha},F,-\xi N^{-\frac{1}{2}},\lambda)\}\right| \le BN^{-3/2},$$

$$(8.19)\qquad \sup_{|x|\le C}\left|P_{F,\mu,\lambda}\left(\{\lambda(1-\lambda)N\int\psi_1^2(t)dt\}^{\frac{1}{2}}(M'_\lambda-\mu) \le x\right) - L^{*}(x)\right| \le BN^{-3/2}.$$

There is no unique natural measure to assess the performance of the estimators M and M'_λ on the basis of the expansions (8.5)-(8.7) and (8.19) and consequently there is no unique natural definition of the deficiency of M with respect to M'_λ either. Let us, for a moment, indicate the dependence on the sample size N in our notation and write M_N and $M'_{\lambda,N}$ for M and M'_λ. For any real ξ we define the deficiency $D_N(\xi)$ of the sequence of estimators $\{M_N\}$ with respect to the estimator $M'_{\lambda,N}$ by equating the d.f.'s of $(M_N+D_N-\mu)$ under $P_{F,\mu,\lambda}$ (or $P_{F,\mu}$) and of $(M'_{\lambda,N}-\mu)$ under $P_{F,\mu,\lambda}$ at the point $\xi N^{-\frac{1}{2}}$, thus

$$(8.20)\qquad P_{F,\mu}(M_{N+D_N}-\mu \le \xi N^{-\frac{1}{2}}) = P_{F,\mu,\lambda}(M'_{\lambda,N}-\mu \le \xi N^{-\frac{1}{2}}),$$

with the usual convention that the probability on the left is defined by linear interpolation for nonintegral values of $N+D_N$. Of course, one will normally not be inclined to judge the performance of $\{M_N\}$ with respect to $M'_{\lambda,N}$ on the basis of $D_N(\xi)$ for one value of ξ only, but rather on the behavior of $D_N(\xi)$ as a function of ξ. In our asymptotic study this will not make any difference because the expansions for $D_N(\xi)$ will be found to be independent of ξ.

Turning to the corresponding tests, we let $d_N(\alpha,\theta)$ denote the deficiency in the usual sense of the locally most powerful rank test (or its approximate scores version) with respect to the locally most powerful test for the problem of testing $H_F = (F,F)$ against $K_{F,\theta,\lambda} = (F(\cdot+\lambda\theta),F(\cdot-(1-\lambda)\theta))$ at level α. Since we shall be concerned with negative as well as positive values of θ, we note that for positive (negative) θ the tests involved reject H_F for large (small) values of the statistics given in (2.2) and (8.10), where the scores in (2.2) are exact or approximate scores generated by $-\Psi_1$.

Let F_1 be given by definition 7.1 and define

$$(8.21)\qquad \tilde{D}_{N,1} = -\tfrac{1}{4}\frac{\int\Psi_1^4(t)dt}{\{\int\Psi_1^2(t)dt\}^2} - \tfrac{3}{4} + \frac{1}{\int\Psi_1^2(t)dt}\sum_{j=1}^{N}\sigma^2(\Psi_1(U_{j:N})),$$

$$\tilde{D}_{N,2} = -\tfrac{1}{4}\frac{\int\Psi_1^4(t)dt}{\{\int\Psi_1^2(t)dt\}^2} - \tfrac{3}{4} + \frac{1}{\int\Psi_1^2(t)dt}\int_{N^{-1}}^{1-N^{-1}}(\Psi_1'(t))^2t(1-t)dt,$$

$$\tilde{D}_{N,3} = -\tfrac{1}{4}\frac{\int\Psi_1^4(t)dt}{\{\int\Psi_1^2(t)dt\}^2} - \tfrac{3}{4} + \frac{1}{\int\Psi_1^2(t)dt}\sum_{j=1}^{N}E\left(\Psi_1(U_{j:N}) - \Psi_1\left(\frac{j}{N+1}\right)\right)^2.$$

THEOREM 8.3. *Let $d_N(\alpha,\theta)$ be the deficiency of the locally most powerful rank test with respect to the locally most powerful test for testing H_F against $K_{F,\theta,\lambda}$ at level α. Let $D_N(\xi)$ be the deficiency of the Hodges-Lehmann estimator associated with the locally most powerful rank test with respect to the maximum likelihood estimator for estimating μ under $K_{F,\mu,\lambda}$.*

Suppose that $F \in F_1$ *and that* f *is st*
positive numbers and suppose that c
exist positive numbers $B, \delta_1, \delta_2, \ldots,$
on F, c, C *and* ε, *such that*

$$|D_N(\xi) - d_N(\tfrac{1}{2}, -\xi N^{-\frac{1}{2}})| \leq \delta_N \tag{8.22}$$

$$|D_N(\xi) - \tilde{D}_{N,1}| \leq \delta_N, \tag{8.23}$$

$$|D_N(\xi) - \tilde{D}_{N,2}| \leq \delta_N + BN^{-\frac{1}{2}} \tag{8.24}$$

If in the locally most powerful ran
the exact scores are replaced by the
then (8.22) *and* (8.24) *remain valid*

$$|D_N(\xi) - \tilde{D}_{N,3}| \leq \delta_N. \tag{8.25}$$

PROOF. Since $F_1 \subset F$, the conditions
and (8.5)-(8.7) and (8.19) provide
estimators considered. Substituting
and proceeding exactly as in the pr
and (8.24) for the estimator associa
test and at (8.24) and (8.25) for i

Turning to the corresponding te
that for negative values of ξ the c
for $d_N(\bar{\alpha}, -\xi N^{-\frac{1}{2}})$ is precisely the sa
(8.25) this computation determines

$$|D_N(\xi) - d_N(\bar{\alpha}, -\xi N^{-\frac{1}{2}})| = o \tag{8.26}$$

For positive ξ, $d_N(\alpha, -\xi N^{-\frac{1}{2}})$ refers
fore to the left-sided tests rather
appear in (8.4) and (8.18). Since th
versions of a test sum to 1 if thei

$$(8.27) \qquad |D_N(\xi) - d_N(1-\bar{\alpha}, -\xi N^{-\frac{1}{2}})| = o(1), \qquad \text{for } c \le \xi \le C.$$

Note that (8.26) and (8.27) hold for exact as well as approximate scores and that the remainder terms are uniformly $o(1)$ for fixed F, c, C and ε.

It remains to show that $\bar{\alpha}$ may be replaced by $\frac{1}{2}$ in (8.26) and (8.27). If we take $\Delta = \lambda$ in the power expansion for the locally most powerful test in lemma 8.1 and compare the result with the power expansion for the most powerful test in lemma 7.2, we see that the terms of orders 1 and $N^{-\frac{1}{2}}$ agree and that in the terms of order N^{-1} only certain coefficients differ. Moreover, for $\Delta = \lambda$ the conditions of lemma 8.1 are identical with those of lemma 7.2. This means that if we replace the most powerful test by the locally most powerful test in theorem 7.1, then the theorem will remain valid if some of the coefficients in $\tilde{d}_{N,0}$ are changed. Thus, under the conditions of theorem 7.1 there exists, for exact as well as approximate scores, an expansion for $d_N(\alpha,\theta)$ with a bounded derivative with respect to α and a remainder term $o(1)$. This statement remains correct for $-CN^{-\frac{1}{2}} \le \theta \le -cN^{-\frac{1}{2}}$ because the power expansions in lemma 8.1 and theorem 5.2 are valid for negative θ too (c.f. the remark in the proof of theorem 8.1) so that the only change in the expansion for $d_N(\alpha,\theta)$ is a change of sign of u_α to account for the switch from the right-sided to the left-sided tests. Noting that $c \le |\xi| \le C$ and that $\bar{\alpha} = \frac{1}{2} + O(N^{-\frac{1}{2}})$ we find that we may indeed replace $\bar{\alpha}$ by $\frac{1}{2}$ in (8.26) and (8.27) without affecting the right-hand side and its uniformity for fixed F, c, C and ε. This proves (8.22) and the theorem. □

A number of comments should be made at this point. First of all we recall remarks (ii) and (iii) in our discussion earlier in this section concerning the choice of Δ for which M and M' should be compared. Suppose we define deficiencies $D'_N(\xi)$ by

$$P_{F,\mu}(|M_{N+D'_N} - \mu| \le \xi N^{-\frac{1}{2}}) = P_{F,\mu,\Delta}(|M'_{\Delta,N} - \mu| \le \xi N^{-\frac{1}{2}})$$

for that value of Δ that minimizes the right-hand side. In view of remark (ii), theorem 8.3 implies that $D'_N(\xi)$ is also asymptotically equivalent to the $\tilde{D}_{N,i}$. Thus our results can be thought of as corresponding exactly to those of ABZ (1976) where deficiencies are defined in terms of a positive

quantile of the symmetrically distributed centered estimators in the one-sample problem. Since the deficiency is asymptotically independent of the value of ξ, we obtain the same answers for deficiencies based on reasonable functionals of the distributions of $N^{\frac{1}{2}}|M-\mu|$ and $N^{\frac{1}{2}}|M'-\mu|$, such as the asymptotic second moment. This agrees with what was found in the one-sample case in ALBERS (1974).

The choice $\Delta = \lambda$ is less obvious in equation (8.20) which defines $D_N(\xi)$. In remark (iii) we pointed out that if we would not choose $\Delta = \lambda + o(N^{-\frac{1}{2}})$, then our results would become essentially more complicated. The first source of trouble is the difference of the significance levels $\bar{\alpha}$ and $\tilde{\alpha}$ given by (8.2) and (8.15). Except in the trivial case where $\int \psi_1^3(t)dt = 0$, we find that $(\tilde{\alpha}-\bar{\alpha})$ is of the order of $N^{-\frac{1}{2}}(\Delta-\lambda)$ and a change of the order of $N^{-\frac{1}{2}}(\Delta-\lambda)$ in the level of significance of one of the two tests produces a change of the same order in its power. Unless $\Delta - \lambda = o(N^{-\frac{1}{2}})$ such an effect is not negligible for our purposes and this means that it would no longer be true that the deficiency for the estimators is asymptotically equivalent in the sense of (8.22) to the deficiency of the parent tests at the same level. In fact a correction term of the order of $N^{\frac{1}{2}}(\Delta-\lambda)$ would have to be introduced in (8.22) to ensure its validity. Note that there is no contradiction here with the fact that in the proof of theorem 8.3 we could change $\bar{\alpha}$ to $\frac{1}{2}$ with impunity, because there we were concerned with the same change of level for both tests simultaneously. A second unpleasant consequence of choosing $\Delta = \Delta^0$ (or even $\Delta = \lambda + bN^{-\frac{1}{2}}$ with b independent of x) would be that the expansions for $D_N(\xi)$ would no longer be independent of ξ. By taking $\Delta = \Delta^0$, we would therefore destroy at one stroke the two most striking features of theorem 8.3.

Next we note that upon formal substitution of $\alpha = \frac{1}{2}$ and $\theta = 0$ the expansions for d_N in theorem 7.1 reduce to the expansions for $D_N(\xi)$ in theorem 8.3. This shows that for every $\xi \neq 0$, $D_N(\xi)$ is nonnegative for sufficiently large N.

In the proof of theorem 8.3 we indicated how one can obtain expansions for the deficiency of the locally most powerful rank test or its approximate scores analogue with respect to the locally most powerful test. At that point there was no need to produce these expansions, but we shall do

so now because they may be of independent interest. The simplest way to describe these results is the following. In the formulation of theorem 7.1 change the words "most powerful test" to "locally most powerful test" and K_{F,θ,Δ_0} to $K_{F,\theta,\lambda}$; change $\tilde{d}_{N,0}$ in (7.8) to

$$(8.28) \qquad \tilde{d}_{N,0} = \frac{1}{48}\left[\frac{4\int\psi_1^4(t)dt}{\{\int\psi_1^2(t)dt\}^2}\{3(u_\alpha^2-1)-2\eta^* u_\alpha\} - 12(u_\alpha^2+3-2\eta^* u_\alpha)\right].$$

With these changes theorem 7.1 holds. When comparing the expansions for d_N in (7.8) with those based on (8.28) we see that the expansions in (7.8) consist of three parts. The term involving $(2u_\alpha-\eta^*)^2$ is due to the fact that comparisons with the most powerful test were made for $\Delta = \Delta_0$ rather than $\Delta = \lambda$ (c.f. the discussion following theorem 7.1). The other terms involving η^{*2} represent the deficiency of the locally most powerful test with respect to the most powerful test for $\Delta = \lambda$. The remaining terms are due to the transition from the locally most powerful test to the two rank tests. All four tests are efficient to second order, i.e. for each pair the deficiency is $o(N^{\frac{1}{2}})$, and the reason for this is that the terms of orders 1 and $N^{-\frac{1}{2}}$ are the same in all four power expansions (c.f. the discussion following theorem 7.1).

We conclude with one example of theorem 8.3. For estimating μ in the normal location model $(\Phi(\cdot+\lambda\mu),\Phi(\cdot-(1-\lambda)\mu))$, the deficiency of either one of the Hodges-Lehmann estimators associated with the normal scores test and with van der Waerden's test with respect to the difference of the sample means is given by

$$(8.29) \qquad D_N(\xi) = \log\log N - \frac{3}{2} + \log 2 + \gamma + o(1),$$

where γ is Euler's constant as in (5.50). Note that this expansion is the same as expansion (7.23) for the deficiency of the normal scores test (or van der Waerden's test) with respect to Student's test for any α.

APPENDIX. EXPANSIC

In this apper
theorem 3.1. The c
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(A.3) $\tilde{p}(x,t)$

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(A.4) $\chi_i(x,t)$ (

and for any funct

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Then elementary b

(A.5) $$\tilde{p}(x,0) = \lambda,\ \tilde{p}_{0,1}(x,0) = -\lambda(1-\lambda)\psi_1(x),\ \tilde{p}_{0,2}(x,0) = \lambda(1-\lambda)(1-2\lambda)\psi_2(x),$$

$$\tilde{p}_{0,3}(x,0) = -\lambda(1-\lambda)\{(1-3\lambda+3\lambda^2)\psi_3(x)-6\lambda(1-\lambda)\psi_1(x)\psi_2(x)$$

$$+ 3\lambda(1-\lambda)\psi_1^3(x)\} ,$$

(A.6) $$|\tilde{p}_{0,1}| \le b_1\chi_1,\ |\tilde{p}_{0,2}| \le b_2(\chi_2+\chi_1^2),\ |\tilde{p}_{0,3}| \le b_3(\chi_3+\chi_2^{3/2}+\chi_1^3) ,$$

$$|\tilde{p}_{0,4}| \le b_4(\chi_4+\chi_3^{4/3}+\chi_2^2+\chi_1^4) ,$$

where $b_1,\ldots,b_4$ are positive constants.

Define $\pi_j = E_H P_j$ as in (3.16).

THEOREM A.1. *Suppose that positive numbers* C, C' *and* ε' *exist such that* $\sum a_j^4 \le CN$, $0 \le \theta \le \varepsilon'$ *and* (4.2) *is satisfied. Then there exists* $B > 0$ *depending only on* C, C' *and* ε' *such that*

(A.7) $$\sum a_j(\pi_j-\lambda) = \lambda(1-\lambda)\{-\theta \sum a_j E_F\psi_1(Z_j)+(1-2\lambda)\frac{\theta^2}{2} \sum a_j E_F\psi_2(Z_j)$$

$$- \frac{\theta^3}{6} \sum a_j E_F[(1-3\lambda+3\lambda^2)\psi_3(Z_j)-6\lambda(1-\lambda)\psi_1(Z_j)\psi_2(Z_j)+3\lambda(1-\lambda)\psi_1^3(Z_j)]\}+M_1,$$

$$|M_1| \le BN^{5/4}\theta^4 ;$$

(A.8) $$\sum a_j^2(\pi_j-\lambda) = \lambda(1-\lambda)\{-\theta \sum a_j^2 E_F\psi_1(Z_j)+(1-2\lambda)\frac{\theta^2}{2} \sum a_j^2 E_F\psi_2(Z_j)\} + M_2,$$

$$|M_2| \le BN^{5/4}\theta^3 ;$$

(A.9) $$\sum a_j^3(\pi_j-\lambda) = -\lambda(1-\lambda)\theta \sum a_j^3 E_F\psi_1(Z_j) + M_3 ,$$

$$|M_3| \le BN^{13/12}\theta^2 ;$$

(A.10) $$\sum a_j^2 E_H(P_j-\lambda)^2 = \lambda^2(1-\lambda)^2\theta^2 \sum a_j^2 E_F\psi_1^2(Z_j) + M_4 ,$$

$$|M_4| \le BN^{5/4}\theta^3 ;$$

(A.11) $$\sigma_H^2(\sum a_j P_j) = \lambda^2(1-\lambda)^2\theta^2\sigma_F^2(\sum a_j\psi_1(Z_j)) + M_5 ,$$

$$|M_5| \le B\{N^2\theta^{24/5} + N\theta^{19/5}[E_F|\sum a_j(\psi_1(Z_j) - E_F\psi_1(Z_j))|^3]^{1/3}$$

$$+ \theta^3\sigma_F(\sum a_j\psi_1(Z_j))\sigma_F(\sum a_j\psi_2(Z_j)) + \theta^4\sigma_F^2(\sum a_j\psi_2(Z_j))\} ;$$

(A.12) $$E_H\left(\frac{\lambda g(X_1)}{h(X_1)} - \lambda\right)^4 \le B\theta^4 ;$$

(A.13) $$\left[\sum\{E_H|P_j - \pi_j|^3\}^{4/9}\right]^{9/4} \le \theta^3\left[\sum\{E_F|\psi_1(Z_j)-E_F\psi_1(Z_j)|^3\}^{4/9}\right]^{9/4} + BN^{9/4}\theta^6.$$

PROOF. Although the proof is very similar to that of theorem A1.1 and the relevant part of corollary A1.1 in ABZ (1976), there are additional complications due to the fact that now $\tilde{p}_{0,2}(x,0) \not\equiv 0$. We begin by noting that the distribution of $\xi(X_1,t)$ under F is that of X_1 under $\lambda F(x) + (1-\lambda)F(x-t)$, so that (4.2) and (A.6) imply the existence of $B_1 > 0$ depending only on C' and such that

(A.14) $$\sup\{E_F|\tilde{p}_{0,i}(X_1,\nu\theta)|^{m_i} : 0 \le \nu \le 1\} \le B_1, \qquad i = 1,\ldots,4,$$

where $m_1 = 6$, $m_2 = 3$, $m_3 = 4/3$, $m_4 = 1$.

Using lemma A1.1 of ABZ (1976) together with $\sum a_j^4 \le CN$ and (A.14), we find that

$$|M_1| \le \frac{\theta^4}{24}\sup\{\sum |a_j|E_F|\tilde{p}_{0,4}(Z_j,\nu\theta)| : 0 \le \nu \le 1\}$$

$$\le \frac{(CN)^{\frac{1}{4}}\theta^4}{24}\sup\{NE_F|\tilde{p}_{0,4}(X_1,\nu\theta)| : 0 \le \nu \le 1\} \le \frac{B_1C^{\frac{1}{4}}}{24}N^{5/4}\theta^4,$$

$$|M_2| \le \frac{\theta^3}{6}\sup\{\sum a_j^2E_F|\tilde{p}_{0,3}(Z_j,\nu\theta)| : 0 \le \nu \le 1\}$$

$$\le \frac{\theta^3}{6}\left(\sum a_j^8\right)^{\frac{1}{4}}\sup\{[NE_F|\tilde{p}_{0,3}(X_1,\nu\theta)|^{4/3}]^{3/4} : 0 \le \nu \le 1\}$$

$$\le \frac{B_1^{\frac{3}{4}}C^{\frac{1}{2}}}{6}N^{5/4}\theta^3 ,$$

$$|M_3| \le \frac{\theta^2}{2} \sup\{\sum |a_j|^3 E_F|\tilde{p}_{0,2}(Z_j,\nu\theta)|: 0 \le \nu \le 1\}$$

$$\le \frac{\theta^2}{2}\left(\sum|a_j|^{9/2}\right)^{2/3}(NB_1)^{1/3} \le \frac{B_1^{1/3}C^{3/4}}{2} N^{13/12}\theta^2 ,$$

$$|M_4| \le \frac{\theta^3}{6} \sup\{\sum a_j^2 E_F[2|\tilde{p}_{0,3}(Z_j,\nu\theta)|$$

$$+ 6|\tilde{p}_{0,1}(Z_j,\nu\theta)\tilde{p}_{0,2}(Z_j,\nu\theta)|]: 0 \le \nu \le 1\}$$

$$\le \frac{\theta^3}{6}\left[2\left(\sum a_j^8\right)^{\frac{1}{4}}(NB_1)^{\frac{3}{4}} + 6\left(\sum a_j^4\right)^{\frac{1}{2}}(NB_1)^{\frac{1}{2}}\right] \le (B_1^{\frac{1}{2}}+B_1^{\frac{3}{4}})C^{\frac{1}{2}}N^{5/4}\theta^3 ,$$

$$E_H\left(\frac{\lambda g(X_1)}{h(X_1)} - \lambda\right)^4 \le \theta^4 \sup\{E_F\tilde{p}_{0,1}^4(X_1,\nu\theta): 0 \le \nu \le 1\} \le B_1^{2/3}\theta^4 ,$$

which proves (A.7) - (A.10) and (A.12). To establish (A.13) we note that

$$|\tilde{p}(Z_j,\theta) - E_F\tilde{p}(Z_j,\theta)| \le \theta|\tilde{p}_{0,1}(Z_j,0) - E_F\tilde{p}_{0,1}(Z_j,0)|$$

$$+ \frac{\theta^2}{2}\int_0^1 2(1-\nu)\{|\tilde{p}_{0,2}(Z_j,\nu\theta)| + E_F|\tilde{p}_{0,2}(Z_j,\nu\theta)|\}d\nu.$$

Hence

$$E_H|P_j-\pi_j|^3 \le \frac{\theta^3}{16} E_F|\psi_1(Z_j)-E_F\psi_1(Z_j)|^3 + 4\theta^6\int_0^1 2(1-\nu)E_F|\tilde{p}_{0,2}(Z_j,\nu\theta)|^3 d\nu,$$

$$\sum\{E_H|P_j-\pi_j|^3\}^{4/9} \le \theta^{4/3}\sum\{E_F|\psi_1(Z_j)-E_F\psi_1(Z_j)|^3\}^{4/9} + 2(B_1+1)N\theta^{8/3} ,$$

and (A.13) follows.

It remains to prove (A.11). We have

$$\tilde{p}(x,t) - \lambda + \lambda(1-\lambda)t\psi_1(x) - \tfrac{1}{2}\lambda(1-\lambda)(1-2\lambda)t^2\psi_2(x)$$

$$= \frac{t^2}{2}\int_0^1 2(1-\nu)(\tilde{p}_{0,2}(x,\nu t) - \tilde{p}_{02}(x,0))d\nu = \frac{t^3}{6}\int_0^1 3(1-\nu)^2\tilde{p}_{0,3}(x,\nu t)d\nu,$$

and as a result

$$(\tilde{p}(x,t) - \lambda + \lambda(1-\lambda)t\psi_1(x) - \tfrac{1}{2}\lambda(1-\lambda)(1-2\lambda)t^2\psi_2(x))^2$$

$$\le |\frac{t^2}{2}\int_0^1 2(1-\nu)(\tilde{p}_{0,2}(x,\nu t)-\tilde{p}_{0,2}(x,0))d\nu|^{6/5}$$

$$\times |\frac{t^3}{6}\int_0^1 3(1-\nu)^2\tilde{p}_{0,3}(x,\nu t)d\nu|^{4/5}$$

$$\le |t|^{24/5}\{|\tfrac{1}{2}\int_0^1 2(1-\nu)(\tilde{p}_{0,2}(x,\nu t)-\tilde{p}_{0,2}(x,0))d\nu|^3$$

$$+ |\frac{1}{6}\int_0^1 3(1-\nu)^2\tilde{p}_{0,3}(x,\nu t)d\nu|^{4/3}\}$$

$$\le |t|^{24/5}\int_0^1 \{|\tilde{p}_{0,2}(x,\nu t)|^3 + |\tilde{p}_{0,2}(x,0)|^3 + |\tilde{p}_{0,3}(x,\nu t)|^{4/3}\}d\nu .$$

Similarly,

$$|\tilde{p}(x,t) - \lambda + \lambda(1-\lambda)t\psi_1(x) - \tfrac{1}{2}\lambda(1-\lambda)(1-2\lambda)t^2\psi_2(x)|^{3/2}$$

$$\le |t|^{21/5}\int_0^1 \{|\tilde{p}_{0,2}(x,\nu t)|^3 + |\tilde{p}_{0,2}(x,0)|^3 + |\tilde{p}_{0,3}(x,\nu t)|^{4/3}\}d\nu .$$

It follows that

$$\sigma_F^2\Big(\sum a_j\{\tilde{p}(Z_j,\theta)+\lambda(1-\lambda)\theta\psi_1(Z_j)-\tfrac{1}{2}\lambda(1-\lambda)(1-2\lambda)\theta^2\psi_2(Z_j)\}\Big)$$

$$\le N\sum a_j^2 E_F(\tilde{p}(X_1,\theta)-\lambda+\lambda(1-\lambda)\theta\psi_1(X_1)-\tfrac{1}{2}\lambda(1-\lambda)(1-2\lambda)\theta^2\psi_2(X_1))^2$$

$$\le 3B_1C^{\frac{1}{2}}N^2\theta^{24/5} ,$$

$$|\mathrm{cov}_F\Big(\sum a_j\{\tilde{p}(Z_j,\theta)+\lambda(1-\lambda)\theta\psi_1(Z_j)-\tfrac{1}{2}\lambda(1-\lambda)(1-2\lambda)\theta^2\psi_2(Z_j)\},\ \sum a_j\psi_1(Z_j)\Big)|$$

$$\le [E_F|\sum a_j\{\tilde{p}(Z_j,\theta)-\lambda+\lambda(1-\lambda)\theta\psi_1(Z_j)-\tfrac{1}{2}\lambda(1-\lambda)(1-2\lambda)\theta^2\psi_2(Z_j)\}|^{3/2}]^{2/3}$$

$$\times [E_F|\sum a_j(\psi_1(Z_j) - E_F\psi_1(Z_j))|^3]^{1/3} \le$$

$$\leq \left[\left(\sum|a_j^3|\right)^{\frac{1}{2}} NE_F|\tilde{p}(X_1,\theta)-\lambda+\lambda(1-\lambda)\theta\psi_1(X_1)-\tfrac{1}{2}\lambda(1-\lambda)(1-2\lambda)\theta^2\psi_2(X_1)|^{3/2}\right]^{2/3}$$

$$\times [E_F|\sum a_j(\psi_1(Z_j) - E_F\psi_1(Z_j))|^3]^{1/3}$$

$$\leq (3B_1)^{2/3}C^{1/4}N\theta^{14/5}[E_F|\sum a_j(\psi_1(Z_j) - E_F\psi_1(Z_j))|^3]^{1/3} ,$$

$$|\mathrm{cov}_F\left(\sum a_j\{\tilde{p}(Z_j,\theta)+\lambda(1-\lambda)\theta\psi_1(Z_j)-\tfrac{1}{2}\lambda(1-\lambda)(1-2\lambda)\theta^2\psi_2(Z_j)\}, \sum a_j\psi_2(Z_j)\right)|$$

$$\leq (3B_1)^{1/2}C^{1/4}N\theta^{12/5}\sigma_F(\textstyle\sum a_j\psi_2(Z_j)) .$$

These inequalities ensure that there exists $B_2 > 0$ depending only on B_1 and C such that

$$|\sigma_H^2\left(\textstyle\sum a_jP_j\right)-\sigma_F^2\left(\sum a_j\{\lambda(1-\lambda)\theta\psi_1(Z_j)-\tfrac{1}{2}\lambda(1-\lambda)(1-2\lambda)\theta^2\psi_2(Z_j)\}\right)|$$

$$\leq B_2\{N^2\theta^{24/5}+N\theta^{19/5}[E_F|\textstyle\sum a_j(\psi_1(Z_j)-E_F\psi_1(Z_j))|^3]^{1/3}$$

$$+ N\theta^{22/5}\sigma_F(\textstyle\sum a_j\psi_2(Z_j))\} .$$

Since $N\theta^{22/5}\sigma_F(\sum a_j\psi_2(Z_j)) \leq N^2\theta^{24/5} + \theta^4\sigma_F^2(\sum a_j\psi_2(Z_j))$, (A.11) follows immediately and the proof of the theorem is complete. □

COROLLARY A.1. *Suppose that* (3.1) *and* (4.3) *hold and that positive numbers* c, C, C', D, ε *and* ε' *exist such that* (3.10), (3.19), (4.2) *and* (4.4) *are satisfied. Let* K, α_i, $\tilde{K}$, $\tilde{\alpha}_i$ *and* η *be defined by* (3.17), (3.18), (4.5), (4.6) *and* (4.7). *Then there exists* $B > 0$ *depending only on* c, C, C', D, ε *and* ε' *such that*

$$\text{(A.15)}\qquad \sup_x|K\left(x - \frac{\Sigma a_j\pi_j}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}}\right) - \tilde{K}(x-\eta)| \leq B\Big\{N^{-5/4}$$

$$+ N^{-\frac{1}{2}}\theta^3\left[\sum\{E_F|\psi_1(Z_j)-E_F\psi_1(Z_j)|^3\}^{4/9}\right]^{9/4}$$

$$+ N^{-3/4}\theta^3\left[\sum\{E_F(\psi_2(Z_j)-E_F\psi_2(Z_j))^2\}^{2/3}\right]^{3/2}\Big\} ,$$

(A.16) $$\theta^2 \frac{|\Sigma a_j E_F \psi_2(Z_j)|}{(\Sigma a_j^2)^{\frac{1}{2}}} \le BN^{-\frac{1}{2}}, \quad \theta \frac{|\Sigma a_j^2 E_F \psi_1(Z_j)|}{\Sigma a_j^2} \le BN^{-\frac{1}{2}}, \quad \frac{|\Sigma a_j^3|}{(\Sigma a_j^2)^{3/2}} \le BN^{-\frac{1}{2}},$$

(A.17) $$\theta^2 \frac{\sigma_F^2(\Sigma a_j \psi_1(Z_j))}{\Sigma a_j^2} \le B\left\{N^{-1} + N^{-\frac{1}{4}}\theta^3\left[\sum\{E_F|\psi_1(Z_j) - E_F\psi_1(Z_j)|^3\}^{4/9}\right]^{9/4}\right\},$$

and all other terms occurring in $\tilde{\alpha}_0,\ldots,\tilde{\alpha}_5$ *are bounded in absolute value by* BN^{-1}.

PROOF. In this proof $O(x)$ will denote a quantity that is bounded by $B_1|x|$ with B_1 depending only on c, C, C', D, ε and ε'.

We begin by noting that (A.16) and the last statement in corollary A.1 are immediate consequences of Hölder's inequality, (3.10), (4.2) and (4.4). Also

(A.18) $$\theta^2\sigma_F^2\left(\sum a_j\psi_1(Z_j)\right) \le 1 + \theta^3\sigma_F^3\left(\sum a_j\psi_1(Z_j)\right)$$

$$\le 1 + \theta^3 E_F \left|\sum a_j(\psi_1(Z_j) - E_F\psi_1(Z_j))\right|^3$$

$$\le 1 + \theta^3\left[\sum |a_j|\{E_F|\psi_1(Z_j) - E_F\psi_1(Z_j)|^3\}^{1/3}\right]^3$$

$$\le 1 + \theta^3\left(\sum a_j^4\right)^{3/4}\left[\sum\{E_F|\psi_1(Z_j) - E_F\psi_1(Z_j)|^3\}^{4/9}\right]^{9/4},$$

and in view of (3.10) and (4.4), this implies (A.17). For later use we note that similarly

(A.19) $$\sigma_F^2\left(\sum a_j\psi_2(Z_j)\right) \le C^{\frac{1}{2}}N^{\frac{1}{2}}\left[\sum\{E_F(\psi_2(Z_j) - E_F\psi_2(Z_j))^2\}^{2/3}\right]^{3/2}.$$

It remains to prove (A.15). Since (A.15) is trivially satisfied for $N < (D/\varepsilon')^2$, we may assume that $0 \le \theta \le \varepsilon'$ so that theorem A.1 applies. Because of (3.1), $\sum a_j\pi_j = \sum a_j(\pi_j - \lambda)$. In view of the bounds obtained above, we can truncate expansions (A.7) and (A.8) to

$$\text{(A.20)}\qquad \sum a_j\pi_j = \lambda(1-\lambda)\{-\theta\sum a_jE_F\psi_1(Z_j)+(1-2\lambda)\frac{\theta^2}{2}\sum a_jE_F\psi_2(Z_j)\}+O(N\theta^3)$$

$$= -\lambda(1-\lambda)\theta\sum a_jE_F\psi_1(Z_j) + O(N\theta^2) = O(N\theta),$$

$$\text{(A.21)}\qquad \sum a_j^2(\pi_j-\lambda) = -\lambda(1-\lambda)\theta\sum a_j^2E_F\psi_1(Z_j) + O(1) = O(N^{\frac{1}{2}}).$$

Using (A.8) - (A.11), (A.20), (A.21), (3.10), (3.19) and (4.4) we expand $\alpha_0,\ldots,\alpha_5$ and find

$$\text{(A.22)}\qquad \sup_x|K(x) - \hat{K}(x)| = O\Big(N^{-5/4+\theta^{19/5}}[E_F|\sum a_j(\psi_1(Z_j) - E_F\psi_1(Z_j))|^3]^{1/3}$$

$$+ N^{-1}\theta^3\sigma_F\Big(\sum a_j\psi_1(Z_j)\Big)\sigma_F\Big(\sum a_j\psi_2(Z_j)\Big) + N^{-1}\theta^4\sigma_F^2\Big(\sum a_j\psi_2(Z_j)\Big)\Big),$$

where

$$\text{(A.23)}\qquad \hat{K}(x) = \Phi(x) - \phi(x)\sum_{k=0}^{5}\hat{\alpha}_kH_k(x)\ ,$$

$$\text{(A.24)}\qquad \hat{\alpha}_0 = -\left(\frac{\lambda(1-\lambda)}{\Sigma a_j^2}\right)^{\frac{1}{2}}N^{-1}\theta\sum a_jE_F\psi_1(Z_j)\ ,$$

$$\hat{\alpha}_1 = \tilde{\alpha}_1 - \frac{1}{8\Sigma a_j^2}\lambda(1-\lambda)(1-2\lambda)^2\theta^4\{\sum a_jE_F\psi_2(Z_j)\}^2\ ,$$

$$\hat{\alpha}_2 = \tilde{\alpha}_2 + \frac{\{\lambda(1-\lambda)\}^{\frac{1}{2}}}{4(\Sigma a_j^2)^{3/2}}(1-2\lambda)^2\theta^3\sum a_j^2E_F\psi_1(Z_j)\sum a_jE_F\psi_2(Z_j)\ ,$$

$$\hat{\alpha}_3 = \tilde{\alpha}_3 - \frac{1}{12(\Sigma a_j^2)^2}(1-2\lambda)^2\theta^2\sum a_j^3\sum a_jE_F\psi_2(Z_j),$$

$$\hat{\alpha}_k = \tilde{\alpha}_k \quad \text{for } k = 4,5\ ,$$

with $\tilde{\alpha}_k$ as given by (4.6). By applying elementary inequalities (A.22) may be simplified to

$$\text{(A.25)}\qquad \sup_x|K(x) - \hat{K}(x)| = O\Big(N^{-5/4+N^{-5/4}\theta^3E_F}|\sum a_j(\psi_1(Z_j) - E_F\psi_1(Z_j))|^3$$

$$+ N^{-5/4}\theta^3\sigma_F^2\Big(\sum a_j\psi_2(Z_j)\Big)\Big)\ .$$

With the aid of (A.7), (A.20) and the bounds obtained in the first part of the proof we now expand $\hat{K}(x-\sum a_j\pi_j\{\lambda(1-\lambda)\sum a_j^2\}^{-\frac{1}{2}})$ about the point $(x-\eta)$ and obtain

$$\text{(A.26)} \qquad \sup_x \left|\hat{K}\left(x - \frac{\Sigma a_j\pi_j}{\{\lambda(1-\lambda)\Sigma a_j^2\}^{\frac{1}{2}}}\right) - \tilde{K}(x-\eta)\right| = O\left(N^{-5/4}+N^{-1}\theta^3\sigma_F^2\left(\sum a_j\psi_1(Z_j)\right)\right)$$

with $\tilde{K}$ as given by (4.5). Combining (A.25), (A.26), (A.18) and (A.19) we see that (A.15) and corollary A.1 are proved. □

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